JELIA 2008

Justification Logic

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This lecture outlook

1. What is Justification Logic?

2. Why do we need Justification Logic?

3. What does Justification Logic offer?

4. Any successes for far?

5. What is next?

Mainstream Epistemology:

Starting point: tripartite approach to knowledge (usu
to Plato)

$$\textit{Knowledge} \quad \sim \quad \textit{Justified True Belief}$$

In the wake of papers by Russell, Gettier, and other
criticized, revised; now is generally regarded as a neces
for knowledge.

Logic of Knowledge: the model-theoretic approach
Hintikka, ...) has dominated modal logic and formal
since the 1960s.

F is known $\sim$ F holds at all possible epistemic

Easy, visual, useful in many cases, but misses the mark
What if F holds at all possible worlds, e.g., a mathe
say $P \neq NP$, but the agent is simply not aware of th
lack of evidence, proof, justification, etc.?

Speaking informally: modal logic offers a limited form

Knowledge $\sim$ True Belief.

There were no justifications in the modal logic of kno
a principal gap between mainstream and formal episte

Obvious defect: Logical Omniscience

A basic principle of modal logic (of knowledge, belief

$$\Box(F \to G) \to (\Box F \to \Box G).$$

At each world, the agent is supposed to "know" all
quences of his/her assumptions.

"Each agent who knows the rules of Chess should
there is a winning strategy for White."

"Suppose one knows a product of two (very large) pr
sense does he/she know each of the primes, given tha
may take billions of years of computation?"

Less visible but more fundamental defect:
failure of epistemic closure

A basic principle of modal logic of knowledge:

$$\Box(F \to G) \to (\Box F \to \Box G).$$

fails to represent the epistemic closure principle

one knows everything that one knows to be impli
by what one knows.

Adding justifications into the language

$$t{:}F$$

t is a justification of F for a given agen

t is accepted by agent as a justification o

t is a sufficient resource for F

F satisfies conditions t

etc.

Basic Justification Logic J, the language

Justification terms are built from *variables* $x, y, z, \ldots$
$a, b, c, \ldots$ by means of operations: '$\cdot$' and '$+$'

x

a

$a\cdot x + b\cdot y$

$z\cdot(a\cdot x + b\cdot y)$, etc.

Formulas: usual, with addition of new constructions

$c{:}(A\wedge B\rightarrow A)$

$x{:}A\rightarrow(c\cdot x){:}B$

$x{:}A\vee y{:}B\rightarrow(a\cdot x + b\cdot y){:}(A\vee B)$, etc.

Basic Justification Logic J

- The standard axioms and rules of classical proposit
- $s{:}F \rightarrow (s+t){:}F, \quad t{:}F \rightarrow (s+t){:}F$
- $t{:}(F \rightarrow G) \rightarrow (s{:}F \rightarrow (t \cdot s){:}G)$

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Sum $s+t$ pools together s and t without performing
action, e.g., chapters - a handbook.

Application $s\cdot t$ performs an elementary epistemic act
conclusions G for all F justified by s and all $F \to G$ jus

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Reflects very basic reasoning about justifications.

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Reflects basic reasoning about justifications.
Justifications are not assumed to be factive.
No logical truths are assumed *a priori* as justified for
Good for conditional statements:

if x is a justification for A, then t(x) is a justifica

Old Epistemic Modal language:
New Justification Logic language:

Introducing some a priori justified knowledge

Reasoning with justifications treats some logical tru
justified. Consider a logical axiom:

$$A \wedge B \rightarrow A$$

To assume it justified, use a constant

$$c{:}(A \wedge B \rightarrow A)$$

This new axiom may also be assumed justified

$$d{:}c{:}(A \wedge B \rightarrow A), \text{etc.}$$

Constant Specifications range from empty (Cartesi
the total (all axioms are justified to any depth) at ou

Internalization (for sufficiently rich constant spec

$$\vdash F \text{ yields } \vdash t{:}F \text{ for some } t.$$

Is the explicit version of the Necessitation Rule in mo

$$\vdash F \text{ yields } \vdash \Box F.$$

Examples of reasoning in J

$A \wedge B \rightarrow A$ - logical axiom

$a{:}(A \wedge B \rightarrow A)$ - constant specification

$a{:}(A \wedge B \rightarrow A) \rightarrow (x{:}(A \wedge B) \rightarrow (a \cdot x){:}A)$ - Application

$x{:}(A \wedge B) \rightarrow (a \cdot x){:}A$ - by *Modus Ponens*

If x *is a justification for* $A \wedge B$ *then* $a \cdot x$ *is a justif*
provided a *is a proof (justification) for the logical axi*

Examples of reasoning in J

$a{:}(A \to A \vee B)$ - constant specification
$x{:}A \to (a\cdot x){:}(A \vee B)$ - by Application and *Modus Ponen*

$b{:}(B \to A \vee B)$ - constant specification
$y{:}B \to (b\cdot y){:}(A \vee B)$ - by Application and *Modus Ponen*

$(a\cdot x){:}(A \vee B) \to (a\cdot x + b\cdot y){:}(A \vee B)$ - by Sum
$(b\cdot y){:}(A \vee B) \to (a\cdot x + b\cdot y){:}(A \vee B)$ - by Sum

$x{:}A \vee y{:}B \to (a\cdot x + b\cdot y){:}(A \vee B)$.

Sum '$+$' is used here to reconcile distinct justification
formula $(a\cdot x){:}(A \vee B)$ and $(b\cdot y){:}(A \vee B)$.

Red Barn Example (Kripke, 1980)

Suppose I am driving through a neighborhood in which
to me, papier-mâché barns are scattered, and I see t
in front of me is a barn. Because I have barn-before
I believe that the object in front of me is a barn.
suggest that I fail to know barn. But now suppose t
borhood has no fake red barns, and I also notice that
front of me is red, so I know a red barn is there. This
being a red barn, which I know, entails there being a
do not, is an embarrassment.

Formalization of *RBE* in the modal epistemic log

B - ‘the object which I see is a barn’

R - ‘the object which I see is red’

$\Box$ is my belief modality.

1. $\Box B$ - this is belief, but not knowledge
2. $\Box(B \wedge R)$ - this is knowledge
3. $(B \wedge R) \rightarrow B$ - logical &-axiom
4. $\Box[(B \wedge R) \rightarrow B]$ - knowledge (&-axiom is assumed t

As we see, *1, 2, and 4 constitute a failure of the r* *the epistemic closure principle.*

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The reason - material implication, which does not re nection between knowledge assertions 2, 4, and 1: RB belief claim 1 which is not related to knowledge asser

RBE in Justification Logic

1. $u{:}B$ - belief, not knowledge, by assumption
2. $v{:}(B \wedge R)$ - belief, which is knowledge, by assumptio
3. $(B \wedge R) \rightarrow B$ - &-axiom
4. $a{:}[(B \wedge R) \rightarrow B]$ - Constant Specification
5. $v{:}(B \wedge R) \rightarrow (a \cdot v){:}B$, by Application.

The paradox disappears! Instead of deriving 1 from have derived $(a \cdot v){:}B$, but not $u{:}B$, i.e., I know B fo NOT for reason u. Note, that 1 remains a case of bel knowledge without creating any contradiction.

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Moral: Justification logic offers a better formalizatic temic closure principle: $s{:}F$ & $t{:}(F \rightarrow G)$ $\rightarrow (t \cdot s){:}G)$

Epistemic models for J (Fitting-style)

Kripke model + **possible evidence function** $\mathcal{E}(t, F)$:

t is a possible evidence for F at world u.

Principal definition $t{:}F$ holds at u iff

1. $v \Vdash F$ whenever uRv (the usual Kripke condition fo
2. t is a possible evidence for F at u.

Soundness and Completeness take place.

Justification Logic vs Epistemic Modal Logic

Epistemic Modal Logic = Justification Logic + Forg

Justification Logic vs Epistemic Modal Logic

Epistemic Modal Logic = Justification Logic + Forg

Justification Logic = Epistemic Modal Logic + F

Realization of K **in** J **(the same holds for other ma
modal logics:** T**,** K4**,** K4D**,** S4**,** K45**,** K45D**,** S5)

1. *The forgetful projection of* J *is* K*-compliant.*

2. *For each theorem* F *of* K*, one can recover a w
polynomial) for each occurrence of* $\Box$ *in* F *in such a
resulting formula* F^r *is derivable in* J*.*

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Realization provides a justification semantics for

F is known $\sim$ *F has an adequate justif*

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Realization provides a non-Kripkean semantics for

$$\Box F \quad \sim \quad \textit{there exists a justification fo}$$

What does Justification Logic offer?

It adds a long-anticipated mathematical notion of jus formal epistemology, making it more expressive. We capacity to reason about justifications, simple and co can compare different pieces of evidence pertaining to We can measure the complexity of justifications, thus c logic of knowledge to a rich complexity theory, etc.

What does Justification Logic offer?

Justification logic provides a novel, evidence-based evidence-tracking which can be a valuable tool for bust justifications from a larger body of justifications necessarily reliable.

Successes so far

Solution to Gödel's problem of the intended provab
for modal logic S4.

Completion of Gödel's draft of the Logic of Proofs (

A faithful formalization of Brouwer-Heyting-Kolmogo
of proofs for intuitionistic logic.

Successes so far

A new take on the logical omniscience problem whic
istic feature of modal epistemic logic that the agent
"know" all logical consequences of his/her assumptio

S.A. & Kuznets (2007) considered logical omniscienc
complexity problem and established that modal-base
of knowledge is indeed logically omniscient, whereas e
presentation of knowledge is not logically omniscient.

Successes so far

Justification logic furnishes a new, evidence-based f
the logic of knowledge, according to which

F is known

is interpreted as

F has an adequate justification

Successes so far

Interesting applications to well-known problems in epist
malization of Gettier, Kripke examples (in this talk), th
Paradox and the Knower Paradox (Dean & Kurokawa

Some interest from Cryptography community.

Successes so far

NSF-level grants in several countries, fast-growing inte
munity of researchers, jobs, students, etc.

What is next?

Knowledge, belief, and evidence are fundamental co significance spans many areas of human activity: cor and artificial intelligence, mathematics, economics anc cryptography, philosophy, and other disciplines. Just promises significant impact on the aforementioned are lar, the capacity to keep track of pieces of evidence, c and select those that are appropriate seems to be a tool.