Justification Logic

by *Sergei Artemov* (Graduate Center CUNY)

Analytic Proof Systems 2, Phnom Penh, November 12, 2006

In This Talk

We overview a general purpose formal propositional **theory of justification** based on the classical logic augmented by justification assertions ***t:F*** interpreted as

a given agent has accepted ***t*** *as a justification for* ***F***

This theory grew from the Logic of Proofs **LP** with its semantics of mathematical proofs.

In This Talk

This theory offers new formal methods for studying justifications, knowledge and beliefs, both proof theoretical (via complete systems of axioms) and semantical (via epistemic Fitting models).

We apply this theory to Plato's definition of *knowledge* as *justified true belief*, and well-known Gettier counter-examples to this definition.

Plato's Dialogue Theaetetus:

Knowledge is Justified True Belief

A subject S knows that a proposition P is true iff:

1. *S believes that P is true,*
2. *P is true,*
3. *S is evidentially justified in believing that P is true.*

It was widely accepted until 1963 when a paper by ***Edmund Gettier*** *(Analysis 23(1963):121-123) provoked widespread attempts to revise or replace it.*

Later Developments

- **Brouwer**: mathematical truth = provability
- **Skolem**: quantifiers = ghosts of functions
- **Kolmogorov**: problem solutions (proofs) have an abstract structure, hence “logic of solutions” and intuitionistic logic
- **BHK**-semantics: informal “proof tables”
- **Gödel**: modal logic of provability,
  the first (incomplete) sketch of the logic of proofs

Later Developments

- **Curry-Howard**:
  Combinatory Logic ≈ Hilbert-style proofs,
  λ-calculus ≈ natural derivations (no iterations yet).
- **Kleene** realizability semantics:
  evidence = computational programs.
- Boolos, Solovay: a complete modal logic of formal provability **GL**; no individual proofs (justifictions) yet.
- **S.A.** - Logic of Proofs **LP** corresponding to Gödel's design

Gettier Example: Case I

Smith has applied for a job, but has a justified belief that 'Jones will get the job.' He also has a justified belief that 'Jones has 10 coins in his pocket.' Smith therefore (justifiably) concludes ... that 'the man who will get the job has 10 coins in his pocket.'

In fact, Jones does not get the job. Instead, Smith does. However, as it happens, Smith also has 10 coins in his pocket. So his belief that 'the man who will get the job has 10 coins in his pocket' was justified and true. But it does not appear to be knowledge.

Gettier Example: Case II

Smith has a justified belief that 'Jones owns a Ford.' Smith therefore (justifiably) concludes ... that 'Jones owns a Ford, or Brown is in Barcelona,' even though Smith has no knowledge whatsoever about the location of Brown.

In fact, Jones does not own a Ford, but by sheer coincidence, Brown really is in Barcelona. Again, Smith had a belief that was true and justified, but not knowledge.

More General Gettier-style Examples

Smith walks into a room and seems to see Jones in it; she immediately forms the justified belief, 'Jones is in the room.' But in fact, it is not Jones that Smith saw; it was a life-size replica propped in Jones's chair. Nevertheless, Jones is in the room; she is just hiding under the desk reading comic books while her replica makes it seem as though she is in. So Smith's belief is not only justified but also true.

Exercise: produce your own Gettier-style example.

Notational convention

Notational convention: $\mathbf{K}_b(F)$ *stands for*

agent *b* knows that *F*

In case of one agent, we use $\mathbf{K}F$ *instead of* $\mathbf{K}_b(F)$.

Justification assertions have a format $t{:}_b(F)$ *that reads as*

t* is accepted by agent *b* as a justification of *F

Again, we say $t{:}F$ *instead of* $t{:}_b(F)$, *for brevity, whenever it is safe.*

Responses to Gettier: "fourth condition"

No false premises

No knowledge can be claimed if it relies on a false belief. So, in Case I, the belief 'Jones will get the job' is false.

Formal representation: ***t:F → F should hold for all t:F's that the knowledge claim relies on.***

Responses to Gettier: "fourth condition"

Goldman's reliabilism:

a subject's belief is justified only if the truth of a belief has caused the subject to have that belief (in the appropriate way);

Formal representation: $\boldsymbol{t{:}F \rightarrow F}$

and for a justified true belief to count as knowledge, the subject must also be able to 'correctly reconstruct' (mentally) that causal chain.

Formal representation: **there should be also a special justification *c* for $\boldsymbol{t{:}F \rightarrow F}$, i.e. $\boldsymbol{c{:}(t{:}F \rightarrow F)}$**

Responses to Gettier:
"fourth condition"

Lehrer-Paxson's undefeasibility condition:

knowledge is undefeated justified true belief - which is to say that a justified true belief counts as knowledge if and only if it is also the case that there is no further truth which, had the subject known it, would have defeated her present justification for the belief.

Formal representation: ***if t:F, then for any other piece of evidence s, a joint evidence t+s is still an evidence for F***

$$t{:}F \rightarrow (t+s){:}F$$

Responses to Gettier: "fourth condition"

Dretske's conclusive reasons, Nozick's truth-tracking

A reason must exist for the belief that would not be true if the belief itself were false.

Formal representation: $\neg F \rightarrow \neg t{:}F$, which is logically equivalent to $t{:}F \rightarrow F$.

If, for example, I believe that there is a chair in front of me, the reason for believing that it is there would not exist if the belief were false (that is, if the chair were not there).

Formal representation: $t{:}F \rightarrow (\neg F \rightarrow \neg t{:}F)$, which is logically equivalent to $t{:}F \rightarrow F$.

Our Response to Gettier

What must be added to* Justified True Belief *in order to have knowledge?

Our Response to Gettier

What must be added to* Justified True Belief *in order to have knowledge?

Answer: ***all the above and more logic!***

Our Response to Gettier

What must be added to* Justified True Belief *in order to have knowledge?

Answer: ***all the above and more logic!***

Justification Logic

Justification Logic is consistent with the effort by Curry, Howard, Gödel, Kolmogorov, Gettier, Goldman, Lehrer, Paxson, Dretske, Nozick, and others.

Plato's ***JTB*** definition of knowledge naturally defines a translation from the language of justifications to Hintikka's modal language of knowledge.

Justification Logic

We show that *Justification Logic* exactly corresponds to Hintikka's *Epistemic Modal Logic* via JTB-translation

Logic of Justifications $\Leftrightarrow$ ***Epistemic Modal Logic***

$\uparrow$

Plato's JTB translation

Preliminary Assumptions

- *Justifications are **abstract objects** which have **structure**. We introduce a set of basic operations on justifications and establish their connection to epistemic modal logic.*
- *The usual **potential executability** assumptions: atomic justifications are feasible in time and space for an agent to inspect and accept; basic operations on justifications are feasible; agent does not loose or forget justifications; agent applies the laws of classical logic and accepts their conclusions; etc.*
- *We consider both: **(partial) justifications** which do not necessarily bring knowledge, and **sufficient justifications** which yield knowledge.*

Basic Principles: Applicability

Application operation *takes justifications* **s** *and* **t** *and produces a justification* **s·t** *such that*

$$\text{if } \mathbf{s}{:}(\mathbf{F}\to\mathbf{G}) \text{ and } \mathbf{t}{:}\mathbf{F}, \text{ then } (\mathbf{s}\cdot\mathbf{t}){:}\mathbf{G}.$$

Symbolically

$$\mathbf{s}{:}(\mathbf{F}\to\mathbf{G}) \wedge \mathbf{t}{:}\mathbf{F} \to (\mathbf{s}\cdot\mathbf{t}){:}\mathbf{G}.$$

It is a basic property of justifications, implicitly assumed by Gettier, as well as by combinatory logic, lambda-calculi, BHK semantics, realizability, etc.

Basic Principles: **Applicability**

The corresponding modal epistemic principle

$$\mathbf{K}(F\to G)\wedge\mathbf{K}F\to\mathbf{K}G\ ,$$

smuggles the *logical omniscience* defect into the modal epistemic logic because the latter does not have the capacity to measure knowledge.

Logic of justifications escapes logical omniscience by keeping track of the size of evidence terms.

Basic Principles: **Monotonicity**

(cf. Lehrer-Paxson's principle)

*If **s:F**, then whatever evidence **t** occurs, the pool **s+t** of **s** and **t** remains a sufficient justification for **F**.*

Operation '**+**' takes justifications ***s*** and ***t*** and produces ***s+t*** which is a justification for everything justified by ***s*** or ***t***.

$$s{:}F \rightarrow (s+t){:}F \quad \text{and} \quad t{:}F \rightarrow (s+t){:}F\ .$$

Basic Principles: Monotonicity

A built-in feature of epistemic modal logics!

Consider a valid modal derivation

$$\textit{from } A\to KC \textit{ and } B\to KC \textbf{ infer } A\vee B\to KC.$$

Then the antecedent assumes $A\to x{:}C$ *and* $B\to y{:}C$ *for some unspecified justifications* **x** *and* **y**. *The above modal rule claims that there is an evidence* **f** *depending on* **x**,**y**, *such that* $A\vee B\to f(x,y){:}C$. *Substitute* $A=x{:}C$, $B=y{:}C$ *to get* $x{:}C \vee y{:}C\to f(x,y){:}C$, *hence*

$$x{:}C\to f(x,y){:}C \quad \text{and} \quad y{:}C\to f(x,y){:}C.$$

It remains only to name **f(x,y)** *as* **x+y**.

Basic Principles: **Logical Awareness**

Logical axioms are justified ex officio

An agent is aware of all logical axioms (including the ones concerning justifications) and accepts them as fully justified.

The natural way of formalizing this principle is introducing an evidence constant ***c*** *for each occurrence of an axiom* ***A*** *and postulating* ***c:A****.*

A similar principle is also widely accepted in modal epistemic logic: for each valid fact ***F*** about knowledge, by the modal rule of necessitation, **K*F*** holds.

Additional Principles: **Sufficiency**

A justification of *F* is sufficient for an agent to conclude that *F* is true

Additional Principles: **Sufficiency**

This is the principal outcome of the Gettier examples,

a partiality of a justification cannot be compensated by the truth condition to create knowledge.

The truth alone, when separated from a knower, does not create knowledge. There should be a vehicle (which we call ***justification****) that connects the realm of truth with the realm of knowledge, and this vehicle should carry* ***the whole truth****, not just a part of it.*

Additional Principles: Sufficiency

Partial justifications do not create knowledge regardless to whether the belief if true. Only complete justifications do.

Smith is running for a position which requires the unanimous support of a committee of ten people in a secret voting. After the first nine ballots came out YES, whether Smith knows that he has been elected?

Perhaps, the answer should be ‘no.’ Changing the size of the Committee to 100 and the number of votes counted to 99 does not seem to alter the outcome here: no knowledge occurs before ALL the ballots are counted.

Additional Principles: **Sufficiency**

Suppose all members of the committee have indeed cast their votes for Smith. Does it alter our judgement of whether Smith had a knowledge of 'Smith is elected' before all the ballots are counted?

In does not look this way; the truth of the proposition 'Smith is elected' has no any bearing on whether Smith has a knowledge of 'Smith is elected' before all the ballots are counted.

Additional Principles: **Sufficiency**

Sufficiency yields the **reflexivity axiom**

$$t{:}F \rightarrow F$$

similar to the epistemic axiom

$$\mathbf{K}F \rightarrow F\ ,$$

which is widely accepted as a basic property of knowledge (Plato, Wittgenstein, Hintikka, Nozick, etc.)

Additional Principles: **Introspection**

One of the fundamental principles of knowledge is identifying

knowing

and

knowing that one knows.

In the formal modal setting this corresponds to the principle

$$\mathbf{K}F \rightarrow \mathbf{KK}F\ .$$

This principle has an adequate explicit counterpart

justified

yields

verifiably justified,

since justifications are assumed to be verifiable.

Additional Principles: **Introspection**

The mere fact that the agent accepts ***t*** as a sufficient evidence of ***F*** serves as a sufficient evidence that ***t:F***. Often, such \`meta-evidence' has a physical form, e.g.,

- a referee report certifying that ***t:F***,
- a record in the captain's log proving that ***t:F***,
- a computer verification output with a proof that ***t:F***,
- a proof that ***t*** is a proof of ***F**, etc.*

Introspection assumes that given ***t*** an agent produces a justification ***!t*** of ***t:F***

$$t{:}F \rightarrow\, !t{:}(t{:}F)$$

Negative introspection $\neg t{:}F \rightarrow\, ?t{:}(\neg t{:}F)$

Summary of Justification Principles

Basic

- **Applicability**
- **Monotonicity**
- **Logical Awareness**

Summary of Justification Principles

Basic

- **Applicability**
- **Monotonicity**
- **Logical Awareness**

Additional

- **Sufficiency**
- **Introspection**
- **Negative Introspection**

Justification terms (polynomials)

Built from variables ***x, y, z , . . .*** and constants ***a, b, c, . . .*** by means of operations

- *application* '$\cdot$'
- *sum* ' **+** '

Constants denote atomic justifications which the system no longer analyzes.

Variables denote unspecified justifications.

Justification terms (polynomials)

More elaborate models could also use additional operations on justifications, e.g.

- *verifier* ‘ **!**’
- *negative verifier* ‘ **?**’

Logic of (Partial) Justifications **J**

- **Classical propositional logic**
- **Applicability** $s{:}(F\rightarrow G)\rightarrow(t{:}F\rightarrow(s\cdot t){:}G)$
- **Monotonicity** $s{:}F\rightarrow(s+t){:}F$, $t{:}F\rightarrow(s+t){:}F$
- **Logical Awareness** *for each axiom* ***A*** *pick a constant* ***c*** *and declare a new axiom* ***c:A****.*

Logic of Sufficient Justifications **SJ**

SJ = J + Reflexivity Axiom $t{:}F \rightarrow F$

Adding Positive Introspection

J4 = J + Introspection $t{:}F \rightarrow\ !t{:}t{:}F$

SJ4 = SJ + Introspection $t{:}F \rightarrow\ !t{:}t{:}F$

Adding Negative Introspection

J45 = J4 + Negative Introspection

$$\neg t{:}F \rightarrow ?t{:}(\neg t{:}F)$$

SJ45 = SJ4 + Negative Introspection

$$\neg t{:}F \rightarrow ?t{:}(\neg t{:}F)$$

Checklist of basic justification systems

Partial justifications:	Sufficient justifications:
J	**SJ**
J4	**SJ4 (=LP)**
J45	**SJ45**

Skolem Functions vs Quantifiers

Skolem: functions are replacement of quantifiers

$$\exists x A(x) \rightarrow \exists y B(y) \approx \forall x \exists y (A(x) \rightarrow B(y)) \approx A(x) \rightarrow B(f(x))$$

for some function ***f(x)***

F is known (F is believed) ≈ there is a justification of F

$$\mathbf{K}F \approx \exists x(x{:}F)\ .$$

Logic of justifications does not have quantifiers but has enough functions to replace quantifiers a la Skolem, e.g.,

$$\mathbf{K}F \rightarrow \mathbf{K}G \approx \exists x(x{:}F) \rightarrow \exists y(y{:}G) \approx$$

$$\approx x{:}F \rightarrow f(x){:}G$$

for some function ***f(x)***

Skolem Functions vs Quantifiers (FMU: five minute university)

Skolem: *functions can replace quantifiers*

$$\exists x A(x) \rightarrow \exists y B(y) \quad \approx \quad A(x) \rightarrow B(f(x)),$$

assuming new axiom $\exists y B(y) \leftrightarrow B(f(x))$ *that defines* $y=f(x)$

Skolemization Algorithm of replacing $\exists$s by functions:

1. *each negative occurrence* $\exists x A(x)$ *replace by* $A(x)$

2. *each positive occurrence* $\exists y B(y)$ *replace by* $B(f(x))$ *for a fresh function* $f(x)$ *depending on variables* x *from* (1); *add new axiom defining* $f(x)$

A quantified formula is equivalent to a quantifier-free Skolem form in a proper extension by definitions.

Formalizing Plato's JTB paradigm

Transformations corresponding to the **JTB paradigm:**

Forgetful projection

$$\mathbf{s{:}F} \quad \Rightarrow \quad \mathbf{K}F$$

(**F** *is justified*) (**F** *is known*)

JTB projection

$$F \wedge s{:}F \quad \rightarrow \quad \mathbf{K}F$$

(**F** *is true*) ∧ (**F** *is justified*) (**F** *is known*)

Since in ***reflexive modal logics*** $F \wedge \mathbf{K}F \leftrightarrow \mathbf{K}F$,
translations ⇨ *and* ➙ *for such systems coincide.*

Consolidated Correspondence Theorem

Forgetful projection		
J	⇨	**K**
SJ	⇨	**T**
J4	⇨	**K4**
SJ4	⇨	**S4**
J45	⇨	**K45**
SJ45	⇨	**S5**

JTB projection		
J	↘	**T**
SJ	↗	
J4	↘	**S4**
SJ4	↗	
J45	↘	**S5**
SJ45	↗	

Modal epistemic logic does not distinguished JTB projections of partial and sufficient justifications !

Sufficient Justifications are just right for knowledge

JTB correspondence between logics of sufficient justifications and knowledge

SJ → T

SJ4 → S4

SJ45 → S5

Correspondence Theorem and Gettier

Modal epistemic logic does not distinguished JTB projections of partial and sufficient justifications ! This is a formal reincarnation of Gettier's problem:

One cannot distinguish between partial and sufficient justifications (the principal issue of Gettier examples) from inside the traditional modal epistemic logic on the basis of JTB paradigm alone.

Our response to Gettier's challenge:

*to count as knowledge, justified true belief should satisfy the logic of justification principles: **applicability**, **monotonicity**, **logical awareness,** and **sufficiency**.*

Properties of Justification Systems

Usual logical

- ***deduction theorem,***
- ***closure under substitutions***

Specific for Justification Logic

- ***internalization***

  if $\vdash \boldsymbol{F}$, *then* $\vdash \boldsymbol{p{:}F}$ *for some evidence term* $\boldsymbol{p}$

  (every established fact is justified)

- ***realization theorem***

Examples of derivations in **K** *and* **J**

Derivation in K

$A \rightarrow A \vee B$

$\mathrm{K}(A \rightarrow A \vee B)$

$\mathrm{K}A \rightarrow \mathrm{K}(A \vee B)$

$B \rightarrow A \vee B$

$\mathrm{K}(B \rightarrow A \vee B)$

$\mathrm{K}B \rightarrow \mathrm{K}(A \vee B)$

$(\mathrm{K}A \vee \mathrm{K}B) \rightarrow \mathrm{K}(A \vee B)$

Examples of derivations in **K** *and* **J**

Derivation in K

$A \rightarrow A \vee B$

$\mathbf{K}(A \rightarrow A \vee B)$

$\mathbf{K}A \rightarrow \mathbf{K}(A \vee B)$

$B \rightarrow A \vee B$

$\mathbf{K}(B \rightarrow A \vee B)$

$\mathbf{K}B \rightarrow \mathbf{K}(A \vee B)$

$(\mathbf{K}A \vee \mathbf{K}B) \rightarrow \mathbf{K}(A \vee B)$

Derivation in J

$A \rightarrow A \vee B$

$a{:}(A \rightarrow A \vee B)$

$x{:}A \rightarrow (a \cdot x){:}(A \vee B)$

Examples of derivations in **K** *and* **J**

Derivation in K	**Derivation in J**
$A \to A \vee B$	$A \to A \vee B$
$\mathrm{K}(A \to A \vee B)$	$a{:}(A \to A \vee B)$
$\mathrm{K}A \to \mathrm{K}(A \vee B)$	$x{:}A \to (a \cdot x){:}(A \vee B)$
$B \to A \vee B$	$B \to A \vee B$
$\mathrm{K}(B \to A \vee B)$	$b{:}(B \to A \vee B)$
$\mathrm{K}B \to \mathrm{K}(A \vee B)$	$y{:}B \to (b \cdot y){:}(A \vee B)$
$(\mathrm{K}A \vee \mathrm{K}B) \to \mathrm{K}(A \vee B)$	

Examples of derivations in **K** *and* **J**

Derivation in K

$A \rightarrow A \vee B$

$\mathrm{K}(A \rightarrow A \vee B)$

$\mathrm{K}A \rightarrow \mathrm{K}(A \vee B)$

$B \rightarrow A \vee B$

$\mathrm{K}(B \rightarrow A \vee B)$

$\mathrm{K}B \rightarrow \mathrm{K}(A \vee B)$

$(\mathrm{K}A \vee \mathrm{K}B) \rightarrow \mathrm{K}(A \vee B)$

Derivation in J

$A \rightarrow A \vee B$

$a{:}(A \rightarrow A \vee B)$

$x{:}A \rightarrow (a \cdot x){:}(A \vee B) \rightarrow (a \cdot x + b \cdot y){:}(A \vee B)$

$B \rightarrow A \vee B$

$b{:}(B \rightarrow A \vee B)$

$y{:}B \rightarrow (b \cdot y){:}(A \vee B) \rightarrow (a \cdot x + b \cdot y){:}(A \vee B)$

Examples of derivations in **K** *and* **J**

Derivation in K

$A \rightarrow A \vee B$

$\mathbf{K}(A \rightarrow A \vee B)$

$\mathbf{K}A \rightarrow \mathbf{K}(A \vee B)$

$B \rightarrow A \vee B$

$\mathbf{K}(B \rightarrow A \vee B)$

$\mathbf{K}B \rightarrow \mathbf{K}(A \vee B)$

$(\mathbf{K}A \vee \mathbf{K}B) \rightarrow \mathbf{K}(A \vee B)$

Derivation in J

$A \rightarrow A \vee B$

$a{:}(A \rightarrow A \vee B)$

$x{:}A \rightarrow (a \cdot x){:}(A \vee B) \rightarrow (a \cdot x + b \cdot y){:}(A \vee B)$

$B \rightarrow A \vee B$

$b{:}(B \rightarrow A \vee B)$

$y{:}B \rightarrow (b \cdot y){:}(A \vee B) \rightarrow (a \cdot x + b \cdot y){:}(A \vee B)$

$(x{:}A \vee y{:}B) \rightarrow (a \cdot x + b \cdot y){:}(A \vee B)$

From Knowledge to Justifications?

Realization Theorem:

Recovers justification terms for all modal knowledge/belief operators

Applications: **Provability Semantics**

t:F ≈ t is a proof of F

Can be made precise by fixing a base theory with self-referential capacities: Peano Arithmetic, Set Theory, etc.

SJ4 (a.k.a. LP) *is sound and complete with respect to the provability interpretation.*

In particular, **SJ4** *captures all valid principles (in its language) about mathematical proofs.*

Fitting Epistemic Semantics

A **Fitting model** *for is a Kripke model with an extra feature: a* ***possible evidence function*** $\mathcal{E}$***(t,u)****, which specifies in advance whether evidence* **t** *is acceptable for a given formula at any given world* ***u****.*

u $\models$ ***t:F*** *iff*

- ***F*** *holds at all worlds accessible from* ***u*** *(traditional Kripke requirement)*
- ***t i*****s an acceptable evidence for *F* in *u* according to** $\mathcal{E}$***(t,u)***

Reminiscent of Halpern-Moses awareness models, but has a justification structure, which is missing in the latter.

Fitting Epistemic Semantics

In Fitting models (like in Kripke models) the accessibility relation

- for **J** is arbitrary
- for **SJ** is reflexive
- for **J4** is transitive
- for **SJ4** is reflexive and transitive
- for **J45** is transitive and Euclidian
- for **SJ45** is reflexive, symmetric, and transitive

Formalizing Gettier Examples, Case I

Interpretation:

- ***A*** *= Jones gets the job,*
- ***B*** *= Smith gets the job*
- ***C*** *= Jones has 10 coins in his pocket*
- ***D*** *= Smith has 10 coins in his pocket*
- ***x*** *= whatever evidence Smith had about* ***A***
- ***y*** *= whatever evidence Smith had about* ***C***
- ***z*** *= implicitly assumed evidence that Smith has about* $A \rightarrow \neg B$ (Gettier did not mention this evidence, but it is needed for his argument)

Formalizing Gettier Examples, Case I

Assumptions:

1. ***x:A*** *(**x** is a justification of 'Jones will get the job')*
2. ***y:C*** *(**y** is a justification of 'Jones has 10 coins in his pocket')*
3. ***A→¬B*** *(if Smith gets he job, then Jones does not)*
4. ***z:(A→¬B)*** *(**z** is a justification of '3').* This assumption is missing from the original Case I, but it is necessary for Gettier reasoning; '3' alone **does not suffice.**
5. ***B*** *(Smith gets the job)*
6. ***D*** *(Smith has 10 coins in his pocket)*

Formalizing Gettier Examples, Case I

Formalized Gettier's reasoning in J + assumptions:

7. $(z{\cdot}x){:}\ \neg B$, *from 1,4, by application*

8. $p{:}(\neg B \to (B \to D))$, *internalization of a tautology*

9. $(z{\cdot}x){:}\neg B \to (p{\cdot}(z{\cdot}x)){:}(B \to D))$, *by application*

10. $(p{\cdot}(z{\cdot}x)){:}(B \to D))$, *from 7,9*

11. $c{:}(C \to (A \to C))$, *a justified axiom*

12. $y{:}C \to (c{\cdot}y){:}(A \to C)$, *by application*

13. $(c{\cdot}y){:}(A \to C)$, *from 2,12*

14. $t(x,y,z){:}[(A \to C) \wedge (B \to D)]$, *for some* $t(x,y,z)$*. This follows from 10 and 12 by an easy* **J***-reasoning*

Formalizing Gettier Examples, Case I

Case I is consistent in J. Its natural **Fitting model** is:

$$\begin{array}{lll} \textit{`belief world'} & \mathbf{2} \bullet & \boldsymbol{A,C} \\ & \quad\uparrow & \\ \textit{`real world'} & \mathbf{1} \bullet & \boldsymbol{B,C,D} \end{array}$$

Evidence function justifies axioms by appropriate constants and $\boldsymbol{\mathcal{E}(x,1) = \mathcal{E}(x,2)=\{A\}}$,

$\boldsymbol{\mathcal{E}(y,1) = \mathcal{E}(y,2)=\{C\}}$, $\quad\quad \boldsymbol{\mathcal{E}(z,1) = \mathcal{E}(z,2)=\{A\rightarrow\neg B\}}$.

Formalizing Gettier Examples, Case I

Case I is consistent in J. Its natural **Fitting model** is:

'belief world' **2** $\bullet$ ***A,C***

$\uparrow$

'real world' **1** $\bullet$ ***B,C,D***

Evidence function justifies axioms by appropriate constants and $\mathcal{E}(x,1) = \mathcal{E}(x,2)=\{A\}$,

$\mathcal{E}(y,1) = \mathcal{E}(y,2)=\{C\}$, $\mathcal{E}(z,1) = \mathcal{E}(z,2)=\{A\rightarrow\neg B\}$.

$1,2 \models x{:}A,\ \ y{:}C,\ \ z{:}(A\rightarrow\neg B),\ \ t(x,y,z){:}[(A\rightarrow C)\wedge(B\rightarrow D)]$,

*e.g., Smith has a **partially justified true belief** that 'whoever gets the job has 10 coins in his pocket'*

Formalizing Gettier Examples, Case I

Case I is consistent in J. Its natural **Fitting model** is:

$$\begin{array}{rl} \text{'belief world'} & \mathbf{2} \bullet \ \boldsymbol{A,C} \\ & \ \ \uparrow \\ \text{'real world'} & \mathbf{1} \bullet \ \boldsymbol{B,C,D} \end{array}$$

Evidence function justifies axioms by appropriate constants and $\mathcal{E}(x,1) = \mathcal{E}(x,2)=\{A\}$,
$\mathcal{E}(y,1) = \mathcal{E}(y,2)=\{C\}$, $\mathcal{E}(z,1) = \mathcal{E}(z,2)=\{A\to\neg B\}$.

If we make it an SJ-model by stipulating reflexivity of the accessibility relation, then it no longer remains a model for Case I, e.g., $1\models\neg x{:}A$

Formalizing Gettier Examples, Case I

Moreover, Case I is inconsistent in SJ.

Here is a derivation of a contradiction there:

1. ***x*:*A*** *(**x** is a justification of ‘Jones will get the job’)*
3. ***A*→¬*B*** *(if Smith gets he job, then Jones does not)*
5. ***B*** *(Smith gets the job)*
6. ***A**, from 1, by reflexivity*
7. **¬*B***, *from 3, 6*

Formalizing Gettier Examples, Case I

Summary

- Case I is formalizable in both the logic of (partial) justification **J** and the logic of sufficient justifications **SJ**
- Its set of assumptions is **consistent in J** and **inconsistent in SJ**
- Gettier's claim holds for partial justifications but does not hold for sufficient justifications. Case I

constitutes Partially Justified True Belief,

but

does not constitute Sufficiently Justified True Belief

Formalizing Gettier Examples, Case II

Interpretation:

- ***F*** *= Jones owns a Ford,*
- ***B*** *= Brown is in Barcelona*
- ***x*** *= whatever evidence Smith had about* ***F***

Assumptions:

1. ***x:F****,*

2. ***¬F****,*

3. ***B***

Formalizing Gettier Examples, Case II

Gettier reasoning in J + assumptions

4. ***c:(F→F ∨B)***, *justified propositional axiom*
5. ***x:F →(c·x):(F ∨B)***, *by application*
6. ***(c·x):(F ∨B)***, *from 1,5*
7. ***F ∨B***, *from 3*

Therefore, according to Gettier, the sentence ***F ∨B*** is both true and justified but not known

Formalizing Gettier Examples, Case II

Here is a natural **Fitting J-model** for Case II

$$\begin{array}{rl} \textit{`belief world'} & \mathbf{2} \bullet \ \boldsymbol{F} \\ & \quad \uparrow \\ \textit{`real world'} & \mathbf{1} \bullet \ \boldsymbol{B} \end{array}$$

Evidence function justifies axioms by appropriate constants and $\boldsymbol{\mathcal{E}(x,1) = \mathcal{E}(x,2) = \{F\}}$

Formalizing Gettier Examples, Case II

Here is a natural **Fitting J-model** for Case II

$$\begin{tikzcd} \text{‘belief world’} \quad \mathbf{2} \bullet F \\ \text{‘real world’} \quad \mathbf{1} \bullet B \arrow[u] \end{tikzcd}$$

Evidence function justifies axioms by appropriate constants and $\mathcal{E}(x,1) = \mathcal{E}(x,2) = \{F\}$

In this model $\mathbf{1, 2} \models x{:}F, \quad F \vee B, \quad (c\cdot x){:}(F \vee B)$,

i.e., Smith has a ***partially justified true belief*** that *‘Jones owns a Ford, or Brown is in Barcelona’*

Formalizing Gettier Examples, Case II

Here is a natural **Fitting J-model** for Case II

'belief world' **2** • ***F***

↑

'real world' **1** • ***B***

Evidence function justifies axioms by appropriate constants and $\mathcal{E}(x,1) = \mathcal{E}(x,2) = \{F\}$

However, if we make this a SJ-model by adding reflexivity of the accessibility relation, then $1 \models \neg x{:}F$ ***and*** $1 \models \neg(c{\cdot}x){:}(F \vee B)$.

Formalizing Gettier Examples, Case II

Case II is inconsistent in SJ.

1. ***x*:*F***,
2. **¬*F***,
3. ***F*,** *by reflexivity from 1.*

Formalizing Gettier Examples, Case II

Summary

- Case I is formalizable in both the logic of (partial) justification **J** and the logic of sufficient justifications **SJ**
- Case II is **consistent in J** and **inconsistent in SJ**
- Gettier's claim holds for partial justifications but does not hold for sufficient justifications. Case II constitutes

Partially Justified True Belief

but

does not constitute Sufficiently Justified True Belief

Formalizing Gettier Examples, A Generalized Case

Interpretation:

- ***R*** *= Jones is in the room,*
- ***x*** *= whatever evidence Smith had about* ***R***

Assumptions:

1. ***x:R****,*
2. ***R***

Here the set of assumptions is consistent with both **J** and **SJ** and the matter of accepting ***x:R*** becomes a matter of interpreting the verbal description of the puzzle.

Formalizing Gettier Examples, A Generalized Case

Proposed solution. According to **SJ**, Smith should **reject** ***x:R*** since this assumption does not comply with the Logical Awareness principle saying that for some justification ***c***,

$$\boldsymbol{c{:}(x{:}R \rightarrow R)}\ .$$

Such a ***c*** should provide a sufficient evidence that ***x:R*** indeed yields ***R***; this ***c*** should convincingly explain why seeing a Mme Tussaud's style figure of Jones was sufficient for Smith to conclude that Jones is in the room.

There is no any indication of such ***c*** *in the puzzle, which rather tells us the opposite: whatever reasons Smith had to conclude that Jones was in the room were not sufficient.*

Other Kinds of Knowledge: Empirical, Perceptual, A Priori, etc.

It remains to be seen to what extent the formal verification theory is useful for analysis of *empirical, perceptual, a priori* types of knowledge. From the logic of justification perspective such knowledge may be considered as justified by ***atomic justifications*** ready to be incorporated onto reasoning with other justifications according to standards of **Applicability, Monotonicity, Logical Awareness,** and **Sufficiency**.

Applications of Justification Logic (so far)

- A complete axiomatization of mathematical proofs by means of the Logic of Proofs **LP** (**SJ4** in our classification). This answered a long standing question discussed by Brouwer, Kolmogorov, and Gödel in the early 1930s.
- **New foundations** for Hintikka epistemic modal logic. According to the **Realization Theorem,** '***F is known'*** can be read as

  'there is a sufficient justification of F.'
- Non-Kripkian "existential" semantics for major modal logics

Applications of Justification Logic (so far)

- A new approach to the **Logical Omniscience Problem**; justification term show how hard is it to obtain a knowledge from initial assumptions.
- A new approach to common knowledge in AI: **justified common knowledge** provides a more efficient alternative here.
- Applications are anticipated in the areas where epistemic modal logic is traditionally used, e.g., Game Theory and Economics, Decision Theory, etc.

Future work

Major **foundational problems** here are the

- *structure of realizations*
- *multi-agent justifications*
- *belief revision*
- *justifications in non-monotonic reasoning.*