On Two Models of Provability

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Gödel's modal logic approach to analyzing provability attracted a great deal of attention and eventually led to two distinct mathematical models. The first is the modal logic GL, also known as the Provability Logic, which was shown in 1979 by Solovay to be the logic of the formal provability predicate. The second is Gödel's original modal logic of provability S4, together with its explicit counterpart, the Logic of Proofs LP, which was shown in 1995 by Artemov to provide an exact provability semantics for S4. These two models complement each other and cover a wide range of applications, from traditional proof theory to λ-calculi and formal epistemology.

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1. Introduction

In his 1933 paper [**79**], Gödel chose the language of propositional modal logic to describe the basic logical laws of provability. According to his approach, the classical logic is augmented by a new unary logical connective (modality) '$\Box$' where $\Box F$ should be interpreted as

F is provable.

> Gödel's treatment of provability as modality in [**79**] has an interesting prehistory. In his letter to Gödel [**185**] of January 12, 1931, John von Neumann actually used formal provability as a modal-like operator B and gave a shorter, modal-style derivation of Gödel's second incompleteness theorem. Von Neumann freely used such modal logic features as the transitivity axiom $B(a) \rightarrow B(B(a))$, equivalent substitution, and the fact that the modality commutes with the conjunction '$\wedge$.'

Gödel's goal was to provide an exact interpretation of intuitionistic propositional logic within a classical logic with the provability operator, hence giving classical meaning to the basic intuitionistic logical system.

According to Brouwer, the founder of intuitionism, truth in intuitionistic mathematics means the existence of a proof. An axiom system for intuitionistic logic was suggested by Heyting in 1930; its full description may be found in the fundamental monographs [**93, 106, 171**]. By IPC, we infer Heyting's intuitionistic propositional calculus. In 1931–34, Heyting and Kolmogorov gave an informal description of the intended proof-based semantics for intuitionistic logic [**91, 92, 93, 107**], which is now referred to as the *Brouwer-Heyting-Kolmogorov* (*BHK*) *semantics.* According to the *BHK*-conditions, a formula is 'true' if it has a proof. Furthermore, a proof of a compound statement is connected to proofs of its parts in the following way:

- a proof of $A \wedge B$ consists of a proof of proposition A and a proof of proposition B,

- a proof of $A \vee B$ is given by presenting either a proof of A or a proof of B,
- a proof of $A \rightarrow B$ is a construction transforming proofs of A into proofs of B,
- falsehood $\perp$ is a proposition which has no proof; $\neg A$ is shorthand for $A \rightarrow \perp$.

From a foundational point of view, it did not make much sense to understand the above 'proofs' as proofs in an intuitionistic system, which those conditions were supposed to specify. So in 1933 ([**79**]), Gödel took the first step towards developing an exact semantics for intuitionism based on **classical provability**. Gödel considered the classical modal logic S4 to be a calculus describing properties of provability in classical mathematics:

(i) *Axioms and rules of classical propositional logic*,

(ii) $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$,

(iii) $\Box F \rightarrow F$,

(iv) $\Box F \rightarrow \Box\Box F$,

(v) *Rule of necessitation*: $\dfrac{\vdash F}{\vdash \Box F}$.

Based on Brouwer's understanding of logical truth as provability, Gödel defined a translation $tr(F)$ of the propositional formula F in the intuitionistic language into the language of classical modal logic, i.e., $tr(F)$ was obtained by prefixing every subformula of F with the provability modality $\Box$. Informally speaking, when the usual procedure of determining classical truth of a formula is applied to $tr(F)$, it will test the provability (not the truth) of each of F's subformulas in agreement with Brouwer's ideas.

> Even earlier, in 1928, Orlov published the paper [**147**] in Russian, in which he considered an informal modal-like operator of provability, introduced modal postulates (ii)–(v), and described the translation $tr(F)$ from propositional formulas to modal formulas. On the other hand, Orlov chose to base his modal system on a type of relevance logic; his system fell short of S4.

From Gödel's results in [**79**], and the McKinsey-Tarski work on topological semantics for modal logic [**130**], it follows that the translation $tr(F)$ provides a proper embedding of the intuitionistic logic IPC into $\mathsf{S4}$, i.e., an embedding of IPC into classical logic extended by the provability operator.

Theorem 1.1 (Gödel, McKinsey, Tarski)**.** IPC *proves* F $\Leftrightarrow$ $\mathsf{S4}$ *proves tr(F).*

Still, Gödel's original goal of defining IPC in terms of classical provability was not reached, since the connection of $\mathsf{S4}$ to the usual mathematical notion of provability was not established. Moreover, Gödel noticed that the straightforward idea of interpreting modality $\Box F$ as F *is provable in a given formal system* T contradicted Gödel's second incompleteness theorem (cf. [**48, 51, 70, 89, 165**] for basic information concerning proof and provability predicates, as well as Gödel's incompleteness theorems).

> Indeed, $\Box(\Box F \rightarrow F)$ can be derived in $\mathsf{S4}$ by the rule of necessitation from the axiom $\Box F \rightarrow F$. On the other hand, interpreting modality $\Box$ as the predicate $\mathsf{Provable}_T(\cdot)$ of formal provability in theory T and F as contradiction, i.e., $0 = 1$, converts this formula into the false statement that the consistency of T is internally provable in T:
>
> $$\mathsf{Provable}_T\left(\ulcorner \mathit{Consis(T)} \urcorner\right) \ .$$
>
> To see this, it suffices to notice that the following formulas are provably equivalent in T:
>
> $$\begin{aligned} &\mathsf{Provable}_T(\ulcorner 0 = 1 \urcorner) \rightarrow (0 = 1) \ , \\ &\neg\mathsf{Provable}_T(\ulcorner 0 = 1 \urcorner) \ , \\ &\mathit{Consis(T)} \ . \end{aligned}$$
>
> Here $\ulcorner \varphi \urcorner$ stands for the Gödel number of φ. Below we will omit Gödel number notation whenever it is safe, for example, we will write $\mathsf{Provable}(\varphi)$ and $\mathsf{Proof}(t, \varphi)$ instead of $\mathsf{Provable}(\ulcorner \varphi \urcorner)$ and $\mathsf{Proof}(t, \ulcorner \varphi \urcorner)$.

The situation after Gödel's paper [**79**] can be described by the following figure where '$\hookrightarrow$' denotes a proper embedding:

$$\mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow ? \hookrightarrow \textit{CLASSICAL PROOFS}\,.$$

In a public lecture in Vienna in 1938 [**80**], Gödel suggested using the format of explicit proofs *t is a proof of F* for interpreting his provability calculus S4, though he did not give a complete set of principles of the resulting logic of proofs. Unfortunately, Gödel's work [**80**] remained unpublished until 1995, when the Gödelian logic of proofs had already been axiomatized and supplied with completeness theorems connecting it to both S4 and classical proofs.

The provability semantics of S4 was discussed in [**48, 51, 56, 81, 108, 117, 121, 133, 138, 140, 141, 145, 157, 158**] and other papers and books. These works constitute a remarkable contribution to this area, however, they neither found the Gödelian logic of proofs nor provided S4 with a provability interpretation capable of modeling the *BHK* semantics for intuitionistic logic. Comprehensive surveys of work on provability semantics for S4 may be found in [**12, 17, 21**].

The Logic of Proofs LP was first reported in 1994 at a seminar in Amsterdam and at a conference in Münster. Complete proofs of the main theorems of the realizability of S4 in LP, and about the completeness of LP with respect to the standard provability semantics, were published in the technical report [**10**] in 1995. The foundational picture now is

$$\mathsf{IPC} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP} \hookrightarrow \textit{CLASSICAL PROOFS}\,.$$

The correspondence between intuitionistic and modal logics induced by Gödel's translation $tr(F)$ has been studied by Blok, Dummett, Esakia, Flagg, Friedman, Grzegorczyk, Kuznetsov, Lemmon, Maksimova, McKinsey, Muravitsky, Rybakov, Shavrukov, Tarski, and many others. A detailed survey of modal companions of intermediate (or superintuitionistic) logics is given in [**60**]; a brief one is in [**61**], Sections 9.6 and 9.8.

Gödel's 1933 paper [**79**] on a modal logic of provability left two natural open problems:

(A) Find a modal logic of Gödel's predicate of formal provability $\mathsf{Provable}(x)$, which appeared to be 'a provability semantics without a calculus.'

(B) Find a precise provability semantics for the modal logic $\mathsf{S4}$, which appeared to be 'a provability calculus without a provability semantics.'

Problem (A) was solved in 1976 by Solovay, who showed that the modal logic GL (a.k.a. G, L, $\mathsf{K4.W}$, PRL) axiomatized all propositional properties of the provability predicate $\mathsf{Provable}(F)$ ([**48, 51, 63, 166, 167**]). The solution to problem (B) was obtained through the Logic of Proofs LP (see above and Section 3).

The provability logic GL is given by the following list of postulates:

(i) *Axioms and rules of classical propositional logic*,

(ii) $\Box(F\to G)\to(\Box F\to\Box G)$,

(iii) $\Box(\Box F\to F)\to\Box F$,

(iv) $\Box F\to\Box\Box F$,

(v) *Rule of necessitation*: $\dfrac{\vdash F}{\vdash \Box F}$.

Models (A) and (B) have quite different expressive capabilities. The logic GL formalizes Gödel's second incompleteness theorem $\neg\Box(\neg\Box\bot)$, Löb's theorem $\Box(\Box F\to F)\to\Box F$, and a number of other meaningful provability principles. However, proofs as objects are not present in this model. LP naturally extends typed λ-calculus, modal logic, and modal λ-calculus ([**14, 15**]). On the other hand, model (A) cannot express Gödel's incompleteness theorem.

Provability models (A) and (B) complement each other by addressing different areas of application. The provability logic GL finds applications in traditional proof theory (cf. Subsection 2.11).

The Logic of Proofs LP targets areas of typed theories and programming languages, foundations of verification, formal epistemology, etc. (cf. Subsection 3.8).

2. Provability Logic

A significant step towards finding a modal logic of formal provability was made by Löb who formulated in [**125**], on the basis of previous work by Hilbert and Bernays from 1939 (see [**94**]), a number of natural modal-style properties of the formal provability predicate and observed that these properties were sufficient to prove Gödel's second incompleteness theorem. These properties, known as the *Hilbert-Bernays-Löb derivability conditions*, essentially coincide with postulates (ii), (iv), and (v) of the above formulation of GL, i.e., with the modal logic K4. Moreover, Löb found an important strengthening of the Gödel theorem. He established the validity of the following *Löb Rule* about formal provability:

$$\frac{\vdash \Box F \to F}{\vdash F}\ .$$

It was later noticed in (cf. [**127**]) that this rule can be formalized in arithmetic, which gave a valid law of formal provability known as *Löb's principle*:

$$\Box(\Box F \to F) \to \Box F\ .$$

This principle provided the last axiom of the provability logic GL, named after Gödel and Löb. Neither Gödel nor Löb formulated the logic explicitly, though they established the validity of the underlying arithmetical principles. Presumably, it was Smiley, whose work [**164**] on the foundations of ethics was the first to consider GL a modal logic.

Significant progress in the general understanding of the formalization of metamathematics, particularly in [**70**], inspired Kripke, Boolos, de Jongh, and others to look into the problem of modal axiomatization of the logic of provability. More specifically, the effort was concentrated on establishing GL's completeness with respect to the formal provability interpretation. Independently, a similar problem in an algebraic context was considered by Magari and his school in Italy (see [**129**]). A comprehensive account of these early developments in provability logic can be found in [**52**].

H. Friedman formulated the question of decidability of the letterless fragment of provability logic as his Problem 35 in [**74**]. This question, which happened to be much easier than the general case, was immediately answered by a number of people including Boolos [**46**], van Benthem, Bernardi, and Montagna. This result was apparently known to von Neumann as early as 1931 [**185**].

2.1. Solovay's completeness theorem

The problem of finding a modal logic of Gödel's predicate of formal provability $\mathsf{Provable}(x)$ was solved in 1976 by Solovay.

Let $*$ be a mapping from the set of propositional letters to the set of arithmetical sentences. We call such a mapping an (arithmetical) *interpretation.* Given a standard provability predicate $\mathsf{Provable}(x)$ in PA, we can extend the interpretation $*$ to all modal formulas as follows:

- $\bot^* = \bot$; $\top^* = \top$;
- $*$ commutes with all Boolean connectives;
- $(\Box G)^* = \mathsf{Provable}(G^*)$.

The Hilbert-Bernays-Löb derivability conditions, together with the validity of Löb's principle, essentially mean that GL is sound with respect to the arithmetical interpretation.

Proposition 2.1. *If* $\mathsf{GL} \vdash X$*, then for all interpretations* $*$*,* $\mathsf{PA} \vdash X^*$*.*

Solovay in [**167**] established that GL is also complete with respect to the arithmetical interpretation. Solovay also showed that the set of modal formulas expressing universally *true* principles of provability was axiomatized by a decidable extension of GL, which is usually denoted by S. The system S has the axioms

- all theorems of GL (a decidable set),
- $\Box X \to X$,

and *modus ponens* as the sole rule of inference.

Theorem 2.1 (Solovay, [**167**]).

(1) $\mathsf{GL} \vdash X$ *iff for all interpretations* $*$, $\mathsf{PA} \vdash X^*$,

(2) $\mathsf{S} \vdash X$ *iff for all interpretations* $*$, X^* *is true.*

For the proof of this theorem in [**167**], Solovay invented an elegant technique of embedding Kripke models into arithmetic. Variants and generalizations of this construction have been applied to obtain arithmetical completeness results for various logics with provability and interpretability semantics. An inspection of Solovay's construction shows that it works for all natural formal theories containing a rather weak *elementary arithmetic* EA. Such robustness allows us to claim that GL is indeed a universal propositional logic of formal provability.

Whether or not Solovay's theorem can be extended to bounded arithmetic theories such as S^1_2 or S_2 remains an intriguing open question. Interesting partial results here were obtained by Berarducci and Verbrugge in [**43**].

Solovay's results and methods opened a new page in the development of provability logic. Several groups of researchers in the USA (Solovay, Boolos, Smoryński), the Netherlands (D. de Jongh, Visser), Italy (Magari, Montagna, Sambin, Valentini), and the former USSR (Artemov and his students), have started to work intensively in this area. An early textbook by Boolos [**48**], followed by Smoryński's [**166**], played an important educational role.

The following uniform version of Solovay's Theorem 2.1.1 was established independently by Artemov, Avron, Boolos, Montagna, and Visser [**3, 4, 49, 135, 175**]:

There is an arithmetical interpretation $*$ *such that for each modal formula* X, $\mathsf{PA} \vdash X^*$ *iff* $\mathsf{GL} \vdash X$.

The main thrust of the research efforts in the wake of Solovay's theorem was in the direction of generalizing Solovay's results to more expressive languages. Some of the problems that have received prominent attention are covered below.

2.2. Fixed point theorem

As an important early result on the application of modal logic to the study of the concept of provability in formal systems, a theorem stands out that was found independently by de Jongh and Sambin, who established that GL has the fixed point property (see [**48, 51, 165, 166**]). The de Jongh-Sambin fixed point theorem is a striking reproduction of Gödel's fixed point lemma in a propositional language free of coding, self-substitution functions, etc.

A modal formula $F(p)$ is said to be *modalized in* p if every occurrence of the sentence letter p in $F(p)$ is within the scope of $\Box$.

Theorem 2.2 (de Jongh, Sambin)**.** *For every modal formula* $F(p)$ *modalized in the sentence letter* p*, there is a modal formula* H *containing only sentence letters from* F*, not containing* p*, and such that* GL *proves*

$$H \ \leftrightarrow\ F(H)\ .$$

Moreover, any two solutions to this fixed-point equation with respect to F *are provably equivalent in* GL*.*

The uniqueness segment was also established by Bernardi in [**44**].

The proof actually provided an efficient algorithm that, given F, calculates its fixed point H. Here are some examples of F's and their fixed points H.

Modal formula $F(p)$	Its fixed point H
$\Box p$	$\top$
$\Box\neg p$	$\Box\bot$
$\neg\Box p$	$\neg\Box\bot$

$\neg\Box\neg p$	$\bot$
$q \wedge \Box p$	$q \wedge \Box q$

Perhaps the most famous fixed point of the above sort is given by the second Gödel incompleteness theorem. Indeed, consider $\neg\Box p$ as $F(p)$. By the above table, the corresponding fixed point H is $\neg\Box\bot$. Hence GL proves

$$\neg\Box\bot \rightarrow \neg\Box(\neg\Box\bot)\ . \qquad (1)$$

Since the arithmetical interpretation of $\neg\Box\bot$ for a given theory T is the consistency formula *Consis(T)*, this yields that (1) represents the formalized second Gödel incompleteness theorem:

If T is consistent, then T does not prove its consistency

and that this theorem is provable in T.

The fixed point theorem for GL allowed van Benthem [**173**] and then Visser [**184**] to interpret the modal μ-calculus in GL. Together with van Benthem's observation that GL is faithfully interpretable in μ-calculus [**173**], this relates two originally disjoint research areas.

2.3. First-order provability logics

The natural problem of axiomatizing first-order provability logic was first introduced by Boolos in [**48, 50**] as the major open question in this area. A straightforward conjecture that the first-order version of GL axiomatizes first-order provability logic was shown to be false by Montagna [**137**]. A final negative solution was given in papers by Artemov [**5**] and Vardanyan [**174**].

Theorem 2.3 (Artemov, Vardanyan). *First-order provability logic is not recursively axiomatizable.*

In particular, Artemov showed that the set of the first-order modal formulas that are true under any arithmetical interpretation is not arithmetical. This proof used Tennenbaum's well-known theorem about the uniqueness of the recursive model of Peano

Arithmetic. Vardanyan showed that the set of first-order modal formulas that are provable in PA under any interpretation is Π^0_2-complete, thus not effectively axiomatizable. Independently but somewhat later, similar results were obtained by McGee in his Ph.D. thesis; they were never published.

Even more dramatically, [**7**] showed that first-order provability logics are sensitive to a particular formalization of the provability predicate and thus are not robustly defined.

The material on first-order provability logic is extensively covered in a textbook [**51**] and in a survey [**63**].

2.4. Intuitionistic provability logic

The question of generalizing Solovay's results from classical theories to intuitionistic ones, such as Heyting arithmetic HA, proved to be remarkably difficult. Visser, in [**175**], found a number of nontrivial principles of the provability logic of HA. Similar observations were independently made by Gargov and Gavrilenko. In [**177**], a characterization and a decision algorithm for the letterless fragment of the provability logic of HA were obtained, thus solving an intuitionistic analog of Friedman's 35th problem.

Theorem 2.4 (Visser, [**177**])**.** *The letterless fragment of the provability logic of* HA *is decidable.*

Some significant further results were obtained in [**65, 95, 96, 97, 177, 180, 182, 183**], but the general problem of axiomatizing the provability logic of HA remains a major open question.

2.5. Classification of provability logics

Solovay's theorems naturally led to the notion of *provability logic for a given theory T relative to a metatheory U*, which was suggested by Artemov in [**3, 4**] and Visser in [**175**]. This logic, denoted $\boldsymbol{PL}_T(U)$, is defined as the set of all propositional principles of provability in T that can be established by means of U. In particular, GL is the provability logic $\boldsymbol{PL}_T(U)$ with $U = T = \mathsf{PA}$, and Solovay's provability logic S from Theorem 2.1.2 corresponds to $T = \mathsf{PA}$ and U's being the set of all true sentences of arithmetic. The problem of describing all provability logics for a given theory T relative to a metatheory U, where T and U range over extensions of Peano arithmetic, has become known as the *classification problem for provability logics.* Each of these logics extends GL and hence can be represented in the form $\mathsf{GL}X$ which is GL with additional axioms X and modus ponens as the sole rule of inference. Within this notational convention, $\mathsf{S}{=}\mathsf{GL}\{\Box p \to p\}$. Consider sentences $F_n = \Box^{n+1}\bot \to \Box^n\bot$, for $n \in \omega$. In [**4, 6, 176**], the following three families of provability logics were found:

$$\mathsf{GL}_\alpha = \mathsf{GL}\{F_n \mid n \in \alpha\},\ \text{where } \alpha \subseteq \omega\ ;$$

$$\mathsf{GL}^-_\beta = \mathsf{GL}\Big\{\bigvee_{n \notin \beta} \neg F_n\Big\},\ \text{where } \beta \text{ is a confinite subset of } \omega\ ;$$

$$\mathsf{S}_\beta = \mathsf{S} \cap \mathsf{GL}^-_\beta,\ \text{where } \beta \text{ is a confinite subset of } \omega\ .$$

The families GL_α, GL^-_β and S_β are ordered by inclusion of their indices, and $\mathsf{GL}_\beta \subset \mathsf{S}_\beta \subset \mathsf{GL}^-_\beta$ for cofinite β.

In [**6**], the classification problem was reduced to finding all provability logics in the interval between GL_ω and S. In [**101**], Japaridze found a new provability logic D in this interval,

$$\mathsf{D} = \mathsf{GL}\{\neg\Box\bot, \Box(\Box p \vee \Box q) \to (\Box p \vee \Box q)\}\ .$$

He showed that D is the provability logic of PA relative to $\mathsf{PA}+$ *formalized ω-consistency of* PA. This discovery produced one more provability logic series,

$$\mathsf{D}_\beta = \mathsf{D} \cap \mathsf{GL}^-_\beta,\ \text{where } \beta \text{ is a confinite subset of } \omega\ ,$$

with $\mathsf{GL}_\beta \subset \mathsf{D}_\beta \subset \mathsf{S}_\beta \subset \mathsf{GL}^-_\beta$ for cofinite β.

The classification was completed by Beklemishev who showed in [**33**] that no more provability logics exist.

Theorem 2.5 (Beklemishev, [**33**]). *All provability logics occur in* GL_α, GL^-_β, S_β, *and* D_β, *for* $\alpha, \beta \subseteq \omega$, *and* β *cofinite.*

The proof of Theorem 2.5 produced yet another provability interpretation of D which was shown to be the provability logic of any Σ_1-sound-but-not-sound theory relative to the set of all true sentences of arithmetic. For more details, see [**21, 33, 41**].

2.6. Provability logics with additional operators

Solovay's theorems have been generalized to various extensions of the propositional language by additional operators having arithmetical interpretations.

One straightforward generalization is obtained by simultaneously considering several provability operators corresponding to different theories. Already in the simplest case of *bimodal provability logic*, the axiomatization of such logics turns out to be very difficult. The bimodal logics for many natural pairs of theories have been characterized in [**34, 35, 59, 101, 166**]. However, the general classification problem for bimodal provability logics for pairs of recursively enumerable extensions of PA remains a major open question.

Bimodal logic has been used to study relationships between provability and interesting related concepts such as the Mostowski operator, and Rosser, Feferman, and Parikh provabilities (see [**124, 160, 161, 166, 178**]). In a number of cases, Solovay-style arithmetical completeness theorems have been obtained. These results have their origin in an important paper by Guaspari and Solovay [**86**] (see also [**166**]). They consider an extension of the propositional modal language by a *witness comparison* operator, thus allowing the formalization of Rosser-style arguments from his well-known proof of the incompleteness theorem [**153**]. Similar logics have since been used in [**57, 58, 64**], for example, in the study of the speed-up of proofs.

2.7. Generalized provability predicates

A natural generalization of the provability predicate is given by the notion of *n-provability* which is, by definition, a provability predicate in the set of all true arithmetical Π_n-sentences. For $n = 0$, this concept coincides with the usual notion of provability. As was observed in [**166**], the logic of each individual n-provability predicate coincides with GL. A joint logic of n-provability predicates for $n = 0, 1, 2, \ldots$ contains the modalities [0], [1], [2], etc. The arithmetical interpretation of a formula in this language is defined as usual, except that we now require, for each $n \in \omega$, that $[n]$ be interpreted as n-provability.

The system GLP introduced by Japaridze [**101, 102**] is given by the following axioms and rules of inference.

(i) *Axioms of* GL *for each operator* $[n]$,

(ii) $[m]\varphi \to [n]\varphi$, *for* $m \leq n$,

(iii) $\langle m\rangle\varphi \to [n]\langle m\rangle\varphi$, *for* $m < n$,

(iv) *Rule modus ponens*,

(v) *Rule* $\varphi \vdash [n]\varphi$.

Theorem 2.6 (Japaridze). GLP *is sound and complete with respect to the n-provability interpretation.*

Originally, Japaridze established in [**101, 102**] the completeness of GLP for an interpretation of modalities $[n]$ as the provability in arithmetic using not more than n nested applications of the ω-rule. Later, Ignatiev in [**99**] observed that Japaridze's theorem holds for the n-provability interpretation. Ignatiev also found normal forms for letterless formulas in GLP which play a significant role in Section 2.11 (where only the soundness of GLP is essential).

2.8. Interpretability and conservativity logics

Interpretability is one of the central concepts of mathematics and logic. A theory X is interpretable in Y iff the language of X

can be translated into the language of Y in such a way that Y proves the translation of every theorem of X. For example, Peano Arithmetic PA is interpretable in Zermelo-Fraenkel set theory ZF. The importance of this concept lies in its ability to compare theories of different mathematical character in different languages, for example, set theory and arithmetic. The notion of interpretability was given a mathematical shape by Tarski in 1953 in [**170**]. There is not much known about interpretability in general. The modal logic approach provides insights into the structure of interpretability in some special situations when X and Y are finite propositional-style extensions of a base theory containing a certain sufficient amount of arithmetic.

Visser, following Švejdar [**168**], introduced a binary modality $A \rhd B$ to stand for the arithmetization of the statement

the theory $T + A$ interprets $T + B$,

where T contains a sufficient amount of arithmetic, and A's and B's are propositional formulas in the language with '$\rhd$.' This new modality emulates provability $\Box F$ by $\neg F \rhd \bot$ and thus is more expressive than the ordinary $\Box$. The resulting *interpretability logic* substantially depends on the basis theory T.

The following logic IL is the collection of some basic interpretability principles valid in all reasonable theories: axioms and rules of GL plus

- $\Box(A \rightarrow B) \rightarrow A \rhd B$,
- $(A \rhd B \wedge B \rhd C) \rightarrow A \rhd C$,
- $(A \rhd C \wedge B \rhd C) \rightarrow (A \vee B) \rhd C$,
- $A \rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$,
- $\Diamond A \rhd A$.

(We assume here that the interpretability modality '$\rhd$' binds stronger than the Boolean connectives.)

For two important classes of theories T, the interpretability logic has been characterized axiomatically.

Let ILP be IL augmented by the principle

$$A \rhd B \ \rightarrow \ \Box(A \rhd B) \ .$$

Theorem 2.7 (Visser, [**179**])**.** *The interpretability logic of a finitely axiomatizable theory satisfying some natural conditions is* ILP.

In particular, the class of theories covered by this theorem includes the arithmetical theories $\mathsf{I}\Sigma_n$ for all $n = 1, 2, 3, \ldots$, the second-order arithmetic ACA_0, and the von Neumann-Gödel-Bernays theory GB of sets and classes.

Let ILM be IL augmented by Montagna's principle

$$A \rhd B \ \rightarrow\ (A \wedge \Box C) \rhd (B \wedge \Box C)\ .$$

The following theorem was established independently in [**159**] and [**42**].

Theorem 2.8 (Shavrkurov, Berarducci)**.** *The interpretability logic of essentially reflexive theories satisfying some natural conditions is* ILM.

In particular, this theorem states that ILM is the interpretability logic for Peano arithmetic PA and Zermelo-Fraenkel set theory ZF.

An axiomatization of the minimal interpretability logic, i.e., of the set of interpretability principles that hold over all reasonable arithmetical theories, is not known. Important progress in this area has been made by Goris and Joosten, who have found new universal interpretability principles (cf. [**84, 105**]). Yet more new interpretability principles have been found recently by Goris; they were discovered using Kripke semantics and later shown sound for arithmetic.

The $\rhd$ modality has a related *conservativity* interpretation, which leads to the conservativity logics studied in [**87, 88, 98**]. Logics of *interpolability* and of *tolerance*, introduced by Ignatiev and Japaridze [**66, 67, 100**], have a related arithmetical interpretation, but a format which is different from that of interpretability logics; see [**63**] for an overview.

An excellent survey of interpretability logic is given in [**181**]; see also [**63**].

2.9. Magari algebras and propositional second-order provability logic

An algebraic approach to provability logic was initiated by Magari and his students [**128, 129, 135, 136**]. The *provability algebra* of a theory T, also called the *Magari algebra of* T, is defined as the set of T-sentences factorized modulo provable equivalence in T and equipped with the usual Boolean operations together with the provability operator mapping a sentence F to $\mathsf{Provable}_T(F)$.

Using the notion of provability algebra, one can impart a provability semantics to a representative subclass of propositional second-order modal formulas, i.e., modal formulas with quantifiers over arithmetical sentences. These are just first-order formulas over the provability algebra. For several years, the questions of decidability of the propositional second-order provability logic and of the first-order theory of the provability algebra of PA remained open (cf. [**20**]). Shavrukov in [**162**] provided a negative solution to both of these questions.

Theorem 2.9 (Shavrukov, [**162**])**.** *The first-order theory of the provability algebra of* PA *is mutually interpretable with the set of all true arithmetical formulas* TA*.*

This result was proved by one of the most ingenious extensions of Solovay's techniques.

2.10. 'True and Provable' modality

A gap between the provability logic GL and S4 can be bridged to some extent by using the *strong provability* modality $\boxdot F$ which is interpreted as

$$(\boxdot F)^* = F^* \wedge \mathsf{Provable}(F^*) \ .$$

The reflexivity principle

$$\boxdot F \rightarrow F$$

is then vacuously provable, hence the strong provability modality is S4-compliant.

This approach has been explored in [**47, 81, 118**], where it was shown independently that the arithmetically complete modal logic of strong provability coincides with Grzegorczyk's logic Grz, which is the extension of $\mathsf{S4}$ by the axiom

$$\Box(\Box(F\rightarrow\Box F)\rightarrow F)\rightarrow F\ .$$

The modality of strong provability has been further studied in [**142, 143**]; it played a significant role in introducing justification into formal epistemology (cf. [**26, 28, 27**]), as well as in the topological semantics for modal logic (cf. surveys [**68, 76**]).

Strong provability also plays a certain foundational role: it provides an exact provability-based model for intuitionistic logic IPC. Indeed, by Grzegorczyk's result from [**85**], Gödel's translation *tr* specifies an exact embedding of IPC into Grz (cf. Theorem 1.1):

$$\mathsf{IPC}\ \textit{proves}\ F \quad \Leftrightarrow \quad \mathsf{Grz}\ \textit{proves}\ tr(F)\ .$$

However, the foundational significance of this reduction for intuitionistic logic is somewhat limited by a nonconstructive meaning of strong provability as 'classically true and formally provable,' which is incompatible with the intended intuitionistic semantics. The aforementioned embedding does not bring us closer to the *BHK* semantics for IPC either. For more discussion on these matters, see [**8, 12, 119**].

2.11. Applications

The methods of modal provability logic are applicable to the study of fragments of Peano arithmetic.

Using provability logic methods, Beklemishev in [**36**] answered a well-known question: what kind of computable functions could be proved to be total in the fragment of PA where induction is restricted to Π_2-formulas without parameters? He showed that these functions coincide with those that are primitive recursive. In general, provability logic analysis substantially clarified the behavior of parameter-free induction schemata.

Later results [**37, 39**] revealed a deeper connection between provability logic and traditional proof-theoretic questions, such as consistency proofs, ordinal analysis, and independent combinatorial principles. In [**39**], Beklemishev gave an alternative proof of Gentzen's famous theorem on the proof of the consistency of PA by transfinite induction up to the ordinal ϵ_0.

In [**38**] (cf. also surveys [**21, 40**]), Beklemishev suggested a simple PA-independent combinatorial principle called *the Worm Principle*, which is derived from Japaridze's polymodal extension GLP of provability logic (cf. Section 2.7). Finite words in the alphabet of natural numbers will be called *worms*. The Worm Principle asserts the termination of any sequence $w_0, w_1, w_2, \ldots$ of worms inductively constructed according to the following two rules. Suppose $w_m = x_0 \ldots x_n$, then

(i) if $x_n = 0$, then $w_{m+1} := x_0 \ldots x_{n-1}$ (the head of the worm is cut away);

(ii) if $x_n > 0$, set $k := \max\{i < n : x_i < x_n\}$ and let $w_{m+1} = x_0 \ldots x_k(x_{k+1} \ldots x_{n-1}(x_n - 1))^{m+1}$ (the head of the worm decreases by one, and the part after position k is appended to the worm m times).

Clearly, the emerging sequence of worms is fully determined by the initial worm w_0. For example, consider a worm $w_0 = 2031$. Then the sequence looks as follows:

$$\begin{aligned}
w_0 &= 2031\\
w_1 &= 203030\\
w_2 &= 20303\\
w_3 &= 20302222\\
w_4 &= 203022212221222122212221\\
w_5 &= 2030(22212221222122212220)^6\\
\ldots
\end{aligned}$$

Theorem 2.10 (Beklemishev, [**38**]).

(1) *For any initial worm w_0, there is an m such that w_m is empty.*

(2) *The previous statement is unprovable in Peano arithmetic* PA*. In fact, Statement 1 is equivalent to the 1-consistency of* PA*.*

For other PA-independent principles, cf. [**169**].

Japaridze used a technique from the area of Provability Logic to investigate fundamental connections between provability, computability, and truth in his work on Computability Logic [**103, 104**].

Artemov's Logic of Proofs (Section 3) with its applications also emerged from studies in Provability Logic.

3. Logic of Proofs

The source of difficulties in the provability interpretation of modality lies in the implicit nature of the existential quantifier $\exists$. Consider, for instance, the reflection principle in PA, i.e., all formulas of type $\mathsf{Provable}(F) \rightarrow F$. By Gödel's second incompleteness theorem, this principle is not provable in PA, since the consistency formula $\mathsf{Con}(\mathsf{PA})$ coincides with a special case of the reflection principle, namely $\mathsf{Provable}(\bot) \rightarrow \bot$. The formula $\mathsf{Provable}(F)$ is $\exists x \mathsf{Proof}(x, F)$ where $\mathsf{Proof}(x, y)$ is Gödel's *proof predicate*

$$x \textit{ is } (\textit{a code of}\,) \textit{ a proof of a formula } (\textit{having code}) \; y.$$

Assuming $\mathsf{Provable}(F)$ does not yield pointing to any specific proof of F, since this x may be a nonstandard natural number which is not a code of any actual derivation in PA.

For proofs represented by explicit terms, the picture is very different. The principle of *explicit reflection* $\mathsf{Proof}(p, F) \rightarrow F$ is provable in PA for each specific derivation p. Indeed, if $\mathsf{Proof}(p, F)$ holds, then F is evidently provable in PA, and so is the formula $\mathsf{Proof}(p, F) \rightarrow F$. Otherwise, if $\mathsf{Proof}(p, F)$ does not hold, then $\neg\mathsf{Proof}(p, F)$ is true and provable, therefore $\mathsf{Proof}(p, F) \rightarrow F$ is also provable.

This observation suggests a remedy: representing proofs by terms t in the proof formula $\mathsf{Proof}(t, F)$ instead of implicit representation of proofs by existential quantifiers in the provability formula $\exists x \mathsf{Proof}(x, F)$. As we have already mentioned, Gödel suggested using the format of explicit proof terms for the interpretation of $\mathsf{S4}$ as early as 1938, but that paper remained unpublished until 1995 ([**80**]). Independently, the study of explicit modal logics was initiated in [**10, 29, 30, 31, 172**]. In modern terminology, the Logic of Proofs is an instance of Gabbay's Labelled Deductive Systems (cf. [**75**]).

Proof polynomials are terms built from *proof variables* $x, y, z, \ldots$ and *proof constants* $a, b, c, \ldots$ by means of three operations: *application* '$\cdot$' (binary), *union* '$+$' (binary), and *proof checker* '!' (unary). The language of *Logic of Proofs* LP is the language of classical propositional logic supplemented by a new rule for building formulas, namely for each proof polynomial p and formula F, there is a new formula $p{:}F$ denoting 'p is a proof of F.' It is also possible to read this language type-theoretically: formulas become types, and $p{:}F$ denotes 'term p has type F.' We assume also that '$t{:}$' and '$\neg$' bind stronger than '$\wedge, \vee$' which, in turn, bind stronger than '$\rightarrow$.'

Axioms and inference rules of LP:

(i) *Axioms of classical propositional logic*

(ii) $t{:}(F \rightarrow G) \rightarrow (s{:}F \rightarrow (t \cdot s){:}G)$ (*application*)

(iii) $t{:}F \rightarrow F$ (*reflection*)

(iv) $t{:}F \rightarrow !t{:}(t{:}F)$ (*proof checker*)

(v) $s{:}F \rightarrow (s+t){:}F, \quad t{:}F \rightarrow (s+t){:}F$ (*sum*)

(vi) *Rule modus ponens*

(vii) $\vdash c{:}A$, *where* A *is from* (i)–(v), *and* c *is a proof constant* (*Rule of constant specification*)

As one can see from the principles of LP, constants denote proofs of axioms. The application operation corresponds to the internalized *modus ponens* rule: for each s and t, a proof $s \cdot t$ is a proof of all formulas G such that s is a proof of $F \rightarrow G$ and t is a proof of F for some F. The union '$s + t$' of proofs s and t is a proof which proves everything that either s or t does. Finally, '!' is interpreted as a universal program for checking the correctness of proofs, which given a proof t, produces a proof that t proves F ([**10, 12**]). In [**13**], it was noted that proof polynomials represent the whole set of possible operations on proofs for a propositional language. It was shown that any operation on proofs which is invariant with respect to a choice of a normal proof system and which can be specified in a propositional language can be realized by a proof polynomial.

In what follows, '$\vdash$' denotes derivability in LP unless stated otherwise. By a *constant specification* $\mathcal{CS}$, we mean a set of formulas $\{c_1{:}A_1, c_2{:}A_2, \ldots\}$ where each A_i is an axiom from (i)–(v) of LP, and each c_i is a proof constant. By default, with each derivation in LP, we associate a constant specification $\mathcal{CS}$ introduced in this derivation by the use of the rule of constant specification.

One of the basic properties of LP is its capability of internalizing its own derivations. The weak form of this property yields the following admissible rule for LP ([**10, 12**]):

$$\textit{if} \vdash F, \textit{ then} \vdash p{:}F \textit{ for some proof polynomial } p\ .$$

This rule is a translation of the well-known necessitation rule of modal logic

$$\frac{\vdash F}{\vdash \Box F}$$

into the language of explicit proofs. The following more general *internalization rule* holds for LP: *if*

$$A_1, \ldots, A_n \vdash B\ ,$$

then it is possible to construct a proof polynomial $t(x_1, \ldots, x_n)$ *such that*

$$x_1{:}A_1, \ldots, x_n{:}A_n \vdash t(x_1, \ldots, x_n){:}B\ .$$

One might notice that the Curry-Howard isomorphism covers only a simple instance of the proof internalization property where all of $A_1, \ldots, A_n, B$ are purely propositional formulas containing no proof terms. For the Curry-Howard isomorphism basics, see, for example, [**78**].

The decidability of LP was established in [**134**]. Kuznets in [**115**] obtained an upper bound Σ_2^p on the satisfiability problem for LP-formulas in M-models. This bound was lower than the known upper bound *PSPACE* on the satisfiability problem in $\mathsf{S4}$ (under the assumption that $\Sigma_2^p \neq \mathit{PSPACE}$). A possible explanation of why LP wins in complexity over $\mathsf{S4}$ is that the satisfiability test for LP is somewhat similar to type checking, i.e., checking the correctness of assigning types (formulas) to terms (proofs), which is known to be relatively easy in classical cases. Milnikel in [**132**] established Π_2^p-completeness of LP for some natural classes of constant specifications, including so-called injective ones, when each constant denotes a proof of not more than one axiom. Π_2^p-hardness for the whole LP remains an open problem.

N. Krupski in [**109, 110**] considered a representative subsystem of LP, rLP, consisting of formulas $t{:}F$ derivable in LP. The system rLP is as expressible as LP itself, since every F derivable in LP is represented in rLP by $t{:}F$ for an appropriate proof term t. A better upper bound (*NP*) for the decision procedure in rLP was found. In addition, the disjunctive property for the original logic of proofs LP was also established:

$$\textit{if } \mathsf{LP} \vdash s{:}F \vee t{:}G, \textit{ then } \mathsf{LP} \vdash s{:}F \textit{ or } \mathsf{LP} \vdash t{:}G.$$

3.1. Arithmetical Completeness

The Logic of Proofs LP is sound and complete with respect to the natural provability semantics. By *proof system* we mean

1. provably in PA decidable predicate $\mathsf{Proof}(x, y)$ that enumerates all theorems of PA, i.e.,

$$\mathsf{PA} \vdash \varphi \quad \text{iff} \quad \mathsf{Proof}(n, \varphi) \ \text{ holds for some } n\ ,$$

2. computable functions $\mathbf{m}(x, y)$, $\mathbf{a}(x, y)$ and $\mathbf{c}(x)$ such that, for all arithmetical formulas φ, ψ and all natural numbers k, n the following holds:

$$\mathsf{Proof}(k, \varphi \to \psi) \wedge \mathsf{Proof}(n, \varphi) \to \mathsf{Prf}(\mathbf{m}(k, n), \psi)$$

$$\mathsf{Proof}(k,\varphi)\rightarrow\mathsf{Proof}(\mathbf{a}(k,n),\varphi)$$

$$\mathsf{Proof}(n,\varphi)\rightarrow\mathsf{Proof}(\mathbf{a}(k,n),\varphi)$$

$$\mathsf{Proof}(k,\varphi)\rightarrow\mathsf{Proof}(\mathbf{c}(k),\mathsf{Proof}(k,\varphi))\ .$$

The class of proof systems includes the Gödelian proof predicate in PA

x is a Gödel number of a derivation in PA *that contains a formula with a Gödel number y*

with the obvious choice of operations $\mathbf{m}(x,y)$, $\mathbf{a}(x,y)$, and $\mathbf{c}(x)$. In particular, $\mathbf{a}(k,n)$ is the concatenation of proofs k and n, and $\mathbf{c}$ is a computable function that given a Gödel number of a proof k, returns the Gödel number $\mathbf{c}(k)$ of a proof, containing formulas $\mathsf{Proof}(k,\varphi)$ for all φ's such that $\mathsf{Proof}(k,\varphi)$ holds.

An arithmetical interpretation $*$ is determined by a choice of proof system as well as an interpretation of proof variables and constants by numerals (denoting proofs), and propositional variables by arithmetical sentences. Boolean connectives are understood in the same way in both LP and PA, and a formula $p{:}F$ is interpreted as an arithmetical formula $\mathsf{Proof}(p^*,F^*)$.

> This kind of provability semantics is referred to as *call-by-value* semantics; it was introduced in [**11**] and used in [**12, 14, 24, 83, 189**]. A more sophisticated *call-by-name* semantics of the language of LP was introduced in [**10**] and used in [**112, 113, 163, 186**]. Under the call-by-name semantics, proof polynomials are interpreted as Gödel numbers of definable provably recursive arithmetical terms. Call-by-value interpretations may be regarded as a special case of call-by-name interpretations since numerals are definable provably recursive arithmetical terms.

For a given constant specification $\mathcal{CS}$, an interpretation $*$ is called a *$\mathcal{CS}$-interpretation* if all formulas from $\mathcal{CS}$ are true under a given $*$. The following arithmetical completeness theorem has been established in [**10**] for the call-by-name semantics, and in [**11**] for the call-by-value semantics (see also articles [**12, 14**]):

Theorem 3.1 (Artemov, [**10, 11**]). *A formula F is derivable in* LP *with a given constant specification $\mathcal{CS}$ iff* $\mathsf{PA} \vdash F^*$*, for any $\mathcal{CS}$-interpretation $*$.*

This theorem stands if one replaces '$\mathsf{PA} \vdash F^*$' by 'F^* holds in the standard model of arithmetic.'

3.2. Realization Theorem

Another major feature of the Logic of Proofs is its ability to realize all S4-derivable formulas by restoring corresponding proof polynomials inside all occurrences of modality. This fact may be expressed by the following realization theorem ([**10, 12**]). We understand *forgetful projection* of an LP-formula F to be a modal formula obtained by replacing all occurrences of $t{:}(\cdot)$ in F by $\Box(\cdot)$.

Theorem 3.2 (Artemov, [**10**]). S4 *is the forgetful projection of* LP.

That the forgetful projection of LP is S4-compliant is a straightforward observation. The converse has been established in [**10, 12**] by presenting an algorithm which substitutes proof polynomials for all occurrences of modalities in a given cut-free Gentzen-style S4-derivation of a formula F, thereby producing a formula F^r derivable in LP. The original realization algorithms from [**10, 12**] were exponential. Brezhnev and Kuznets in [**55**] offered a realization algorithm of S4 into LP which is polynomial in the size of a cut-free derivation in S4. The lengths of realizing proof polynomials can be kept quadratic in the length of the original cut-free S4-derivation.

Here is an example of an S4-derivation realized as an LP-derivation in the style of the realization theorem 3.2. There are two columns in the table below. The first is a Hilbert-style S4-derivation of a modal formula $\Box A \vee \Box B \to \Box(\Box A \vee B)$. The second column displays corresponding steps of an LP-derivation of a formula

$$x{:}A \vee y{:}B \to (a{\cdot}!x + b{\cdot}y){:}(x{:}A \vee B)$$

with constant specification

$$\{\ a{:}(x{:}A\rightarrow x{:}A\vee B),\ \ b{:}(B\rightarrow x{:}A\vee B)\ \}\ .$$

	Derivation in S4	Derivation in LP
1.	$\Box A\rightarrow\Box A\vee B$	$x{:}A\rightarrow x{:}A\vee B$
2.	$\Box(\Box A\rightarrow\Box A\vee B)$	$a{:}(x{:}A\rightarrow x{:}A\vee B)$
3.	$\Box\Box A\rightarrow\Box(\Box A\vee B)$	$!x{:}x{:}A\rightarrow(a\cdot!x){:}(x{:}A\vee B)$
4.	$\Box A\rightarrow\Box\Box A$	$x{:}A\rightarrow!x{:}x{:}A$
5.	$\Box A\rightarrow\Box(\Box A\vee B)$	$x{:}A\rightarrow(a\cdot!x){:}(x{:}A\vee B)$
$5'$.		$(a\cdot!x){:}(x{:}A\vee B)\rightarrow(a\cdot!x+b\cdot y){:}(x{:}A\vee B)$
$5''$.		$x{:}A\rightarrow(a\cdot!x+b\cdot y){:}(x{:}A\vee B)$
6.	$B\rightarrow\Box A\vee B$	$B\rightarrow x{:}A\vee B$
7.	$\Box(B\rightarrow\Box A\vee B)$	$b{:}(B\rightarrow x{:}A\vee B)$
8.	$\Box B\rightarrow\Box(\Box A\vee B)$	$y{:}B\rightarrow(b\cdot y){:}(x{:}A\vee B)$
$8'$.		$(b\cdot y){:}(x{:}A\vee B)\rightarrow(a\cdot!x+b\cdot y){:}(x{:}A\vee B)$
$8''$.		$y{:}B\rightarrow(a\cdot!x+b\cdot y){:}(x{:}A\vee B)$
9.	$\Box A\vee\Box B\rightarrow\Box(\Box A\vee B)$	$x{:}A\vee y{:}B\rightarrow(a\cdot!x+b\cdot y){:}(x{:}A\vee B)$

Extra steps $5'$, $5''$, $8'$, and $8''$ are needed in the LP case to reconcile different internalized proofs of the same formula: $(a\cdot!x){:}(x{:}A\vee B)$ and $(b\cdot y){:}(x{:}A\vee B)$. The resulting realization respects Skolem's idea that negative occurrences of existential quantifiers (here over proofs hidden in the modality of provability) are realized by free variables whereas positive occurrences are realized by functions of those variables.

Switching from the provability format to the language of specific witnesses reveals hidden self-referentiality of modal logic, i.e., the necessity of using proof assertions of the form $t{:}F(t)$, where t occurs in the very formula $F(t)$ of which it is a proof. A recent result by Kuznets in [**55**] shows that self-referentiality is an intrinsic feature of the modal logic approach to provability in general.

Theorem 3.3 (Kuznets, [**55**]). *Self-referential constant specifications of the sort $c:A(c)$ are necessary for realization of the modal logic* S4 *in the Logic of Proofs* LP.

In particular, the S4-theorem

$$\neg\Box\neg(S \to \Box S)$$

cannot be realized in LP without self-referential constant specifications of the sort $c{:}A(c)$.

Systems of proof polynomials for other classical modal logics K, $\mathsf{K4}$, D, $\mathsf{D4}$, T were described in [**53, 54**]. The case of $\mathsf{S5} = \mathsf{S4} + (\neg\Box F \to \Box\neg\Box F)$ was special because of the presence of negative information about proofs and its connections to formal epistemology. The paper by Artemov, Kazakov, and Shapiro [**24**] introduced a system of proof terms for $\mathsf{S5}$, and established realizability of the logic $\mathsf{S5}$ by these terms, decidability, and completeness of the resulting logic of proofs. An alternative approach, not connected to the arithmetical provability, to representing negative information in the logic of proofs was considered in [**126**].

3.3. Fitting models

The original idea of epistemic semantics for LP can be traced back to Mkrtychev and Fitting. It consists of augmenting Boolean or Kripke models with an *evidence function*, which assigns 'admissible evidence' terms to a statement before deciding its truth value.

Fitting models are defined as follows. A *frame* is a structure (W, R), where W is a non-empty set of *possible worlds* and R is a binary reflexive and transitive *evidence accessibility* relation on W. Given a frame (W, R), a *possible evidence* function $\mathcal{E}$ is a mapping from worlds and proof polynomials to sets of formulas. We can read $F \in \mathcal{E}(u, t)$ as

'F is one of the formulas for which
t serves as possible evidence in world u.'

An evidence function respects the intended meanings of the operations on proof polynomials, i.e., for all proof polynomials s and t, for all formulas F and G, and for all $u, v \in W$, each of the following hold:

(i) *Monotonicity*: uRv implies $\mathcal{E}(u, t) \subseteq \mathcal{E}(v, t)$;

(ii) *Closure*

- *Application*: $F \to G \in \mathcal{E}(u,s)$ and $F \in \mathcal{E}(u,t)$ implies $G \in \mathcal{E}(u, s{\cdot}t)$;
- *Inspection*: $F \in \mathcal{E}(u,t)$ implies $t{:}F \in \mathcal{E}(u, !t)$;
- *Sum*: $\mathcal{E}(u,s) \cup \mathcal{E}(u,t) \subseteq \mathcal{E}(u, s+t)$.

A model is a structure $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$ where (W,R) is a frame, $\mathcal{E}$ is an evidence function on (W,R), and $\Vdash$ is an arbitrary mapping from sentence variables to subsets of W. Given a model $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$, the forcing relation $\Vdash$ is extended from sentence variables to all formulas by the following rules. For each $u \in W$:

(i) $\Vdash$ respects Boolean connectives ($u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$, $u \Vdash \neg F$ iff $u \nVdash F$, etc.);

(ii) $u \Vdash t{:}F$ iff $F \in \mathcal{E}(u,t)$ and $v \Vdash F$ for every $v \in W$ with uRv.

We consider the modality $\Box$, associated with the evidence accessibility relation R. In this terms, the last item of the above definition can be recast as

(ii′) $u \Vdash t{:}F$ iff $u \Vdash \Box F$ and t is an admissible evidence for F at u.

Mkrtychev models are Fitting models with singleton W's. LP was shown to be sound and complete with respect to both Mkrtychev models ([**134**]) and Fitting models ([**72, 73**]). Fitting models were adapted for a multi-agent epistemic setting in [**16, 26, 27, 71**] and became the standard semantics for epistemic modal logics with justification.

In his recent paper [**83**], Goris showed that LP is sound and complete with respect to the call-by-value semantics of proofs in Buss's weak arithmetic S^1_2, thus showing that explicit knowledge can be realized by *PTIME*-computable operations on proofs in a natural mathematical system. Note that the corresponding question for the Provability Logic GL remains a major open problem (cf. Subsection 2.1).

3.4. Joint logics of proofs and provability

The problem of finding a joint logic of proofs and provability has been a natural next step, since there are important principles formulated in a mixed language of formal provability and explicit proofs. For example, the modal principle of negative introspection $\neg\Box F \to \Box\neg\Box F$ is not valid in the provability semantics; neither is a purely explicit version of negative introspection $\neg(x{:}F) \to t(x){:}\neg(x{:}F)$. However, a mixed explicit-implicit principle $\neg(t{:}F) \to \Box\neg(t{:}F)$ is valid in the standard provability semantics.

The joint system of provability and explicit proofs without operations on proof terms, system B, was found in [**9**]. This system describes those principles that have a pure logical character and do not depend on any specific operations of proofs.

The postulates of B consist of those of GL together with the following new principles:

A1. $t{:}F \to F$,

A2. $t{:}F \to \Box t{:}F$,

A3. $\neg t{:}F \to \Box\neg t{:}F$,

RR. *Rule of reflection*: $\dfrac{\vdash \Box F}{\vdash F}$.

Theorem 3.4 (Artemov, [**9**]). B *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

The problem of joining two models of provability, GL and LP, into one model can be specified as that of finding an arithmetically complete logic containing postulates of both GL and LP and closed under internalization.

The first solution to this problem was offered by Yavorskaya (Sidon) in [**163, 186**] who found an arithmetically complete system of provability and explicit proofs, LPP, containing both GL and LP. Along with natural extensions of principles and operations from GL and LP, LPP contains additional operations '$\Uparrow$' and '$\Downarrow$' which were used to secure the internalization property of LPP.

The operation '$\Uparrow$' given a proof t of F, returns a proof $\Uparrow t$ of $\mathsf{Provable}(F)$. The operation '$\Downarrow$' takes a proof t of $\mathsf{Provable}(F)$ and returns a proof $\Downarrow t$ of F. The set of postulates of LPP consists of those of GL and LP together with A2, A3, and RR from B, plus two new principles:

A4. $t{:}F \rightarrow (\Uparrow t){:}\Box F$,

A5. $t{:}\Box F \rightarrow (\Downarrow t){:}F$.

Finally, Nogina in [**26, 144**] noticed that each specific instance of operations '$\Uparrow$' and '$\Downarrow$' can be eliminated, and introduced an arithmetically complete logic GLA joining GL and LP in their original languages. The system GLA is presented in [**26, 144**] by the set of postulates of GL and LP augmented by the principles:

- $t{:}F \rightarrow \Box F$,
- $\neg t{:}F \rightarrow \Box\neg t{:}F$,
- $t{:}\Box F \rightarrow F$.

and *Rule of reflection* RR.

Theorem 3.5.

(1) (Yavorskaya (Sidon), [**163, 186**]) LPP *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

(2) (Nogina, [**26, 144**]) GLA *is sound and complete with respect to the semantics of proofs and provability in Peano arithmetic.*

It was the system GLA which served in [**26, 27**] as a prototype of basic logic of knowledge with justification (cf. Subsection 3.8).

3.5. Quantified logics of proofs

The arithmetical provability semantics for the logic of proofs may be naturally generalized to first-order language and to the language of LP with quantifiers over proofs. Both possibilities of enhancing the expressive power of LP were investigated and in both cases, axiomatizability questions have been answered negatively.

Theorem 3.6.

(1) (Artemov, Yavorskaya (Sidon), [**32**]) *The first-order logic of proofs is not recursively enumerable.*

(2) (Yavorsky, [**190**]) *The logic of proofs with quantifiers over proofs is not recursively enumerable.*

An interesting decidable fragment of the first-order logic of the standard proof predicate was found in [**189**].

3.6. Intuitionistic logic of proofs

As in the case of Provability Logic, a natural question is that of efficient axiomatization of the logic of proofs for Heyting Arithmetic HA. However, unlike the Provability Logic case, the first layer of problems here has a definite resolution. Let us consider so-called intuitionistic basic logic of proofs iBLP where no specific operations on proofs are in the langauge.

The first thing to notice is that in addition to the principles borrowed from the classical Logic of Proofs, there is a principle of decidability of proof assertions

$$t{:}F \vee \neg t{:}F\ .$$

Another source of new principles is the set of admissible propositional rules in HA. As was noticed by Iemhoff, for each admissible rule F/G in HA there is a logic of proofs principle

$$x{:}F \rightarrow G\ .$$

A complete decidable axiomatization iBLP was found by Artemov and Iemhoff in [**22, 23**] with the use of the ideas and technique of Ghilardi. The next natural goal in this direction is the establishment of the arithmetical completeness of intuitionistic logic of proofs with operations corresponding to all admissible rules in HA, cf. [**22, 23**] for precise formulations.

3.7. The logic of single conclusion proofs

By definition, each single conclusion proof, also known as *functional proofs*, proves a unique formula. In the functional logic of proofs, a formula $t{:}F$ still has the meaning 't is a proof of formula F,' but the class of its interpretations is limited to functional proof systems only. It is easy to see that single conclusion proofs lead to modal identities inconsistent with any normal modal logic, e.g., $x{:}\top \rightarrow \neg x{:}(\top \wedge \top)$ is a valid principle of the functional proofs which, however, has the forgetful projection $\Box\top \rightarrow \neg\Box(\top\wedge\top)$ which is incompatible with any normal modal logic.

The mathematical problem here was to give a full axiomatization of all resulting tautologies in the language of LP (without the operation '$+$,' which does not work on functional proofs); this problem was solved by V. Krupski in [**112**].

The functionality property of proofs, which states that if $p{:}F \wedge p{:}G$, then F and G must coincide syntactically, does not look like a propositional condition, since it operates with the strong notion of syntactic coincidence. An adequate propositional description of this property was found in [**29**] using so-called *conditional unification*. It was then generalized in [**112, 113**] to the full language of the logic of proofs.

Each formula C of type $t_1{:}F_1 \wedge \ldots \wedge t_n{:}F_n$ generates a set of quasi-equations of type $S_C := \{\, t_i = t_j \Rightarrow F_i = F_j \mid 1 \leq i, j \leq n \,\}$. A *unifier* σ of S_C is a substitution σ such that either $t_i\sigma \not\equiv t_j\sigma$ or $F_i\sigma \equiv F_j\sigma$ holds for any i, j. Here and below '$X \equiv Y$' denotes the syntactic equality of X and Y. $A = B\,(mod\, S)$ means that for each unifier σ of system S, the property $A\sigma \equiv B\sigma$ holds. This conditional unification was shown to be decidable in the cases under consideration (cf. [**29, 112, 113**]). By *Unification Axiom* we understand the schema

$$t_1{:}F_1 \wedge \ldots \wedge t_n{:}F_n \rightarrow (A \leftrightarrow B)$$

for each condition C of type $t_1{:}F_1 \wedge \ldots \wedge t_n{:}F_n$ and each A, B such that $A = B\,(mod\, S_C)$.

The Logic of Functional Proofs FLP was introduced by V. Krupski in [**112**]. The language of FLP is the language of LP without the operation "+" and without proof constants. The axioms and rules of FLP are:

A0. *Axiom and rules of classical propositional logic*

A1. $t{:}(F \rightarrow G) \rightarrow (s{:}F \rightarrow (t \cdot s){:}G)$

A2. $t{:}F \rightarrow F$

A3. $t{:}F \rightarrow !t{:}t{:}F$

A4. *Unification axiom.*

Theorem 3.7 (V. Krupski, [**112, 113**])**.** *The logic* FLP *is decidable, sound, and complete with respect to the arithmetical provability interpretation based on functional proof predicates.*

The logic of functional proofs was further developed in [**114**], where its extension with references FLP_{ref} was introduced. System FLP_{ref} extends FLP with second-order variables which denote the operation of reconstructing an object from its reference, e.g., determining a formula proven by a given derivation. FLP_{ref} may be also viewed as a natural formal system for admissible inference rules in arithmetic. See also follow-up articles [**156, 187**].

3.8. Applications

Here we will list some conceptual applications of the Logic of Proofs.

1. *Existential semantics for modal logic.* Proof polynomials and LP represent an exact *existential semantics* for mainstream modal logic. Initially, Gödel regarded the modality $\Box F$ as the provability assertion, i.e.,

there exists a proof for F.

Thus, according to Gödel, modality is a Σ_1-sentence, i.e., the one which consists of an existential quantifier (here over proofs) followed by a decidable condition. Such an understanding of modality is typical of 'naive' semantics for a wide range of epistemic and provability logics. Nonetheless, before LP was discovered, major modal logics lacked a mathematical semantics of an existential character. The exception to the rule is the arithmetical provability interpretation for the Provability Logic GL, which still cannot be extended to the major modal logics S4 and S5.

Almost 30 years after the first work by Gödel on the subject, a semantics of a *universal* character was discovered for modal logic, namely Kripke semantics. Modality in that semantics is read informally as the sentence:

In each possible situation, F holds.

Such a reading of modality naturally appears in dynamic and temporal logics aimed at describing computational processes, states of which usually form a (possibly branching) Kripke structure. Universal semantics has been playing a prominent role in modal logic. However, it is not the only possible semantical tool in the study and application of modality. The existential semantics of realizability by proof polynomials can also be useful for foundations and application of modal logic. For more discussion on the existential semantics for modal logic, see [**18**].

2. *Justification Logic.* A major area of application of the Logic of Proofs is epistemology. Books [**69, 131**] serve as excellent introductions to the mathematical logic of knowledge.

Plato's celebrated tripartite definition of knowledge as *justified true belief* is generally regarded in mainstream epistemology as a set of necessary conditions for the possession of knowledge. Due to Hintikka, the 'true belief' components have been fairly formalized by means of modal logic and its possible worlds semantics. The remaining 'justification' condition has received much attention in epistemology (cf., for example, [**45, 77, 82, 90, 120, 122, 123, 146**]), but lacked formal representation. The issue of finding a formal epistemic logic with justification has also been discussed in [**172**]. Such a logic should contain assertions of the form $\Box F$ (*F is known*), along with those of the form $t{:}F$ (*t is a justification for F*). Justification was introduced into formal epistemology in [**16, 26, 27, 28**] by combining Hintikka-style epistemic modal logic with justification calculi arising from the Logic of Proofs LP. Epistemic logic with justification was used in [**16, 19**] to offer a new approach to *common knowledge.* A new modal operator $J\varphi$ for *justified knowledge* introduced in [**16, 19**] is defined as a forgetful projection of justification assertions $t{:}\varphi$ in a multi-agent epistemic logic with common justification. Justified knowledge was shown to be a lighter, constructive version of common knowledge. In particular, in [**2**] it was shown that for a typical epistemic problem, common knowledge systems are conservative over those of justified knowledge, hence whenever the former work, the latter can be used, too. This line of research is picking up rapidly, cf. also [**71, 116, 148, 151, 152, 154, 155, 188**].

3. *Tackling the logical omniscience problem.* The traditional Hintikka-style modal logic approach to knowledge has the well-known defect of *logical omniscience*, which is the unrealistic feature that an agent knows all logical consequences of his/her assumptions ([**69, 139, 149, 150**]). Epistemic systems with justification address the issue of logical omniscience in a natural way. A justified knowledge $t{:}F$ cannot be asserted without presenting an explicit justification t for F, hence justified knowledge does not lead to logical omniscience. This property was formally established in [**25**], where it was shown that Justification Logic is logically omniscient w.r.t. the usual knowledge represented by Hintikka-style epistemic modalities '*F is known*' (modulo common complexity assumptions), and is not logically omniscient w.r.t. the evidence-based knowledge '*t is a justification for F.*'

4. *Reflection in typed combinatory logic and λ-calculus.* Typed λ-calculi and Combinatory Logic are mathematical prototypes of functional programming languages with types (cf., for example, [**62**]). There are reasons to believe that this area would benefit from extending λ-calculi and Combinatory Logic by self-referential capacities which enable systems to simultaneously operate with related objects of different abstraction level: functions, their high level programs, their low level codes, etc. Reflexive Combinatory Logic RCL ([**17**]) was invented to meet these kinds of expectations. RCL introduces a reflexivity mechanism into Combinatory Logic,

hence to λ-calculus. RCL has the implicative intuitionistic (minimal) logic as a type system, a rigid typing. Reflexive combinatory terms are built from variables, 'old' combinators $\mathbf{k}$ and $\mathbf{s}$, and new combinators $\mathbf{d}$, $\mathbf{o}$, and $\mathbf{c}$. The principles of RCL are

A1. $t{:}A\to A$
A2. $\mathbf{k}{:}(A\to(B\to A))$
A3. $\mathbf{s}{:}[(A\to(B\to C))\to((A\to B)\to(A\to C))]$
A4. $\mathbf{d}{:}(t{:}A\to A)$
A5. $\mathbf{o}{:}[u{:}(A\to B)\to(v{:}A\to(u\cdot v){:}B)]$
A6. $\mathbf{c}{:}(t{:}A\to\,!t{:}t{:}A)$

Rule modus ponens,

$$\frac{A\to B \quad A}{B}\ .$$

RCL has a natural provability semantics inherited from LP. Combinatory terms stand for proofs in PA or in intuitionistic arithmetic HA. Formulas $t{:}F$ are interpreted as arithmetical statements about provability, $\mathsf{Proof}(t,F)$, combinators $\mathbf{k}$, $\mathbf{s}$, $\mathbf{d}$, $\mathbf{o}$, and $\mathbf{c}$ denote terms corresponding to proofs of arithmetical translations of axioms A2–A6.

RCL evidently contains implicative intuitionistic logic, ordinary Combinatory Logic $\mathsf{CL}_{\to}$, and is closed under the combinatory application rule

$$\frac{u{:}(A\to B) \quad v{:}A}{(u{\cdot}v){:}B}\ .$$

Furthermore, RCL enjoys the internalization property ([**17**]): if $A_1,\ldots,A_n\vdash B$ then for any set of variables $x_1,\ldots,x_n$ of respective types, it is possible to construct a term $t(x_1,\ldots,x_n)$ such that

$$x_1{:}A_1,\ldots,x_n{:}A_n\vdash t(x_1,\ldots,x_n){:}B\ .$$

It is interesting to consider the following natural (though so far informal) computational semantics for combinators of RCL. This semantics is based on the standard set-theoretic semantics of types, i.e., a type is a set and the implication type $U\to V$ is a set of functions from U to V. Some elements of a given type may be constructive objects which have *names*, i.e., computational programs. Terms of RCL are names of constructive objects, some of them specific, e.g., combinators **k**, **s**, **d**, **o**, or **c**). The type $t{:}F$ is interpreted as a set consisting of the object corresponding to term t.

Basic combinators of RCL are understood as follows. Combinators **k** and **s** have the same meaning as in the combinatory logic $\mathsf{CL}_{\rightarrow}$. For example, **k** maps an element $x \in A$ into the constant function $\lambda y.x$ with y ranging over B. The *denotate* combinator $\mathbf{d} : [t : F \rightarrow F]$ realizes the function which maps a name (program) into the object with the given name. A primary example is the correspondence between indexes of computable functions and functions themselves. The *interpreter* combinator $\mathbf{o} : [u : (F \rightarrow G) \rightarrow (v : F \rightarrow (u \cdot v) : G)]$ realizes the program which maps program u and input v into the result of applying u to v. The *coding* combinator $\mathbf{c}{:}[t{:}F{\rightarrow}!t{:}(t{:}F)]$ maps program t into its code $!t$ (alias, specific key in a database, etc.).

In the followup papers [**109, 111**], N. Krupski established that typability and type restoration can be done in polynomial time and that the derivability relation for RCL is decidable and *PSPACE*-complete.

In [**1**], some version of reflexive λ-calculus was considered that has an unrestricted internalization property.

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