Explicit Modal Logic

Sergei N. Artemov *

May, 1998

Abstract

In 1933 Gödel introduced a modal logic of provability ($\mathcal{S}4$) and left open the problem of a formal provability semantics for this logic. Since then numerous attempts have been made to give an adequate provability semantics to Gödel's provability logic with only partial success. In this paper we give the complete solution to this problem in the Logic of Proofs ($\mathcal{LP}$). $\mathcal{LP}$ implements Gödel's suggestion (1938) of replacing formulas "F is provable" by the propositions for explicit proofs "t is a proof of F" ($t:F$). $\mathcal{LP}$ admits the reflection of explicit proofs $t:F \rightarrow F$ thus circumventing restrictions imposed on the provability operator by Gödel's second incompleteness theorem. $\mathcal{LP}$ formalizes the Kolmogorov calculus of problems and proves the Kolmogorov conjecture that intuitionistic logic coincides with the classical calculus of problems.

Introduction

In 1932 Kolmogorov ([16]) gave an informal description of the *calculus of problems* in classical mathematics and conjectured that it coincides with intuitioinistic propositional logic $\mathcal{I}nt$. Kleene realizability [15], Medvedev finite problems [23] and its variants ([36], [37]) are regarded (cf. [34],[10],[36],[37]) as formalizations of Kolmogorov's calculus of problems. However, they give only necessary conditions for $\mathcal{I}nt$, each of them realizes some formulas not derivable in $\mathcal{I}nt$.

In 1933 Gödel ([12]) defined $\mathcal{I}nt$ on the basis of the notion of *proof* in a classical mathematical system, where "proof" may be regarded as a special case of Kolmogorov's "problem solution". Namely, Gödel introduced the *logic of provability* (coinciding with the modal logic $\mathcal{S}4$) and constructed a conservative embedding of $\mathcal{I}nt$ into $\mathcal{S}4$. $\mathcal{S}4$ has all axioms and rules of classical logic in the modal propositional language along with the axioms $\Box F \rightarrow F$, $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$, $\Box F \rightarrow \Box\Box F$, and the *necessitation rule* $F \vdash \Box F$. In [12] no formal provability semantics for $\mathcal{S}4$ was suggested. The straightforward interpretation of $\Box F$ as the arithmetical formula *Provable*(F)

"there exists a number x which is the code of a proof of F".

leads to logics of formal provability incompatible with $\mathcal{S}4$ (cf.[7],[8]).

*627 Rhodes Hall, Cornell University, Ithaca NY, 14853 U.S.A. `email:artemov@hybrid.cornell.edu`; Moscow University, Russia.

> Let us consider the first order arithmetic $\mathcal{PA}$. If F is the boolean constant *false*, then the $\mathcal{S}4$-axiom $\Box F \rightarrow F$ becomes a statement *Consis* $\mathcal{PA}$, expressing the consistency of $\mathcal{PA}$. By necessitation, $\mathcal{S}4$ derives $\Box(\Box F \rightarrow F)$. The latter formula expresses the assertion that *Consis* $\mathcal{PA}$ is provable in $\mathcal{PA}$, which contradicts the Second Gödel Incompleteness Theorem.

The issue of a provability model for $\mathcal{S}4$ was studied by Gödel [13], Lemmon [20], Myhill [27],[28], Kripke [18], Montague [26], Mints [25], Kuznetsov & Muravitskii [19], Goldblatt [14], Boolos [7],[8] Shapiro [30],[31], Buss [9], Artemov [1], and many others. However, the problem of a formal provability semantics for $\mathcal{S}4$ has remained open.

A principal difficulty here is caused by the existential quantifier over proofs in $Provable(F)$. Indeed, the interpretation of the formula $\Box(\Box F \rightarrow F)$ is

"it is provable that "$Provable(F)$ implies F" "

Provability in $\mathcal{PA}$ can be characterized as "true in all models of $\mathcal{PA}$", including the nonstandard ones. In a given model of $\mathcal{PA}$ an element that instantiates the variable x from the existential quantifier for the code of a proof of F in $Provable(F)$ may be nonstandard. In such a case $Provable(F)$ is true in this model, but there is no "real" $\mathcal{PA}$-derivation behind such an x. So, $\mathcal{PA}$ is not able to conclude that F is true from $Provable(F)$ is true since the latter formula does not necessarily deliver a proof of F.

This consideration suggests replacing the provability formula $Provable(F)$ by the formula for proofs *Proof(t,F)* and the existential quantifier on proofs in the former by Skolem style operations on proofs in the latter. Such a conversion helps avoid evaluation of proofs by nonstandard numbers. Some of this operations come from the proof of Gödel's second incompleteness theorem. Within that proof it was established that

$$\mathcal{PA} \vdash Provable(F \rightarrow G) \wedge Provable(F) \ \rightarrow \ Provable(G).$$

This formula is a "forgetful" version of the following theorem.

> *For some computable function* $m(x, y)$
>
> $$\mathcal{PA} \vdash Proof(s, F \rightarrow G) \wedge Proof(t, F) \ \rightarrow \ Proof(m(s,t), G).$$

A similar decoding can be done for another lemma from Gödel's second incompleteness theorem $\mathcal{PA} \vdash Provable(F) \rightarrow Provable(Provable(F))$.

> *For some computable function* $c(x)$
>
> $$\mathcal{PA} \vdash Proof(t, F) \rightarrow Proof(c(t), Proof(t, F)).$$

In his Lecture at Zilsel's, 1938, (published in 1995 in [13], see also [29]) Gödel sketched a constructive version of $\mathcal{S}4$ with the basic proposition "t is a proof of F" and operations similar to $m(x,y)$ and $c(x)$. This Gödel's suggestion suffices to justify the reflexivity principle along with the necessitation rule. However, the question about a complete set of axioms for a logic of proofs, as well as the question about its ability to realize the entire $\mathcal{S}4$ has remained unanswered. It turned out that Gödel's sketch of 1938 lacks the operation "+", without which a realization of $\mathcal{S}4$ cannot be completed (see Comment 6.6).

In this paper we implement Gödel's suggestion of 1938 and find the *Logic of Proofs* ($\mathcal{LP}$)[1] in the propositional language with an extra basic proposition $t : F$ for "t is a proof of F". We find axiom systems for $\mathcal{LP}$ in Hilbert and Gentzen format, establish its soundness and completeness with respect to the standard provability semantics. We establish that $\mathcal{LP}$ realizes the entire $\mathcal{S}4$ by assigning explicit proof terms to the modalities in every $\mathcal{S}4$-derivation. This yields a positive solution to the problem of finding the intended provability semantics for $\mathcal{S}4$ which in turn proves Kolmogorov's conjecture of 1932 that intuitionistic logic $\mathcal{I}nt$ is nothing but the calculus of problems for systems based on classical logic.

Among the related works there is [11], where Gabbay's Labelled Deductive Systems may serve as a natural framework for $\mathcal{LP}$. The Logic of Proofs may also be regarded as a basic epistemic logic with explicit justifications; a problem of finding such systems was raised by van Benthem in [6]. Intuitionistic Type Theory by Martin-Löf [21], [22] also makes use of the format $t{:}F$ with its informal provability reading.

1 Logic of Proofs

1.1 Definition. The language of Logic of Proofs ($\mathcal{LP}$) contains

the usual language of classical propositional logic
proof variables $x_0, \ldots, x_n, \ldots$, proof constants $a_0, \ldots, a_n, \ldots$
functional symbols: monadic !, binary $\cdot$ and $+$
operator symbol of the type "*term : formula*".

We will use $a, b, c, \ldots$ for proof constants, $u, v, w, x, y, z, \ldots$ for proof variables, i, j, k, l, m, n for natural numbers. Terms are defined by the grammar

$$p ::= x_i \mid a_i \mid !p \mid p_1 \cdot p_2 \mid p_1 + p_2$$

We call these terms *proof polynomials* and denote them by $p,r,s,t\ldots$. By analogy we refer to constants as coefficients. Constants correspond to proofs of a finite fixed set of propositional schemas.

[1] The Logic of Proofs $\mathcal{LP}$ was found by the author independently of Gödel's paper [13]. The first presentations of $\mathcal{LP}$ took place at the author's talks at the conferences in Münster and Amsterdam in 1994. Preliminary versions of $\mathcal{LP}$ along with the completeness theorem, realization of $\mathcal{S}4$ and λ-calculi in $\mathcal{LP}$ appeared in Technical Reports [4] and [5]. Note that despite its title the paper [3] does not introduce $\mathcal{LP}$.

Using t to stand for any term and S for any propositional letter, the formulas are defined by the grammar

$$\sigma ::= S \mid \sigma_1 \rightarrow \sigma_2 \mid \sigma_1 \wedge \sigma_2 \mid \sigma_1 \vee \sigma_2 \mid \neg\sigma \mid t:\sigma$$

The intended semantics for $p:F$ is "p is a proof of F", which will be formalized in the next section. Note that a proof system which provides a formal semantics for $p:F$ is not necessarily deterministic, i.e. p may be a proof of several different F's.

1.2 Definition. The system $\mathcal{LP}_0$. Axioms:

A0. Axioms of classical propositional logic in the language of $\mathcal{LP}$
A1. $t:F \rightarrow F$ *"verification"*
A2. $t:(F \rightarrow G) \rightarrow (s:F \rightarrow (t\cdot s):G)$ *"application"*
A3. $t:F \rightarrow\ !t:(t:F)$ *"proof checker"*
A4. $s:F \rightarrow (s+t):F, \quad t:F \rightarrow (s+t):F$ *"choice"*

Rule of inference:

R1. $$\frac{\Gamma \vdash F \rightarrow G \qquad \Gamma \vdash F}{\Gamma \vdash G}$$ *"modus ponens".*

The system $\mathcal{LP}$ is $\mathcal{LP}_0$ plus the rule

R2. if $\mathbf{A}$ *is an axiom A0 – A4, and c a proof constant, then* $\vdash c:\mathbf{A}$ *"necessitation"*

A *Constant Specification* ($\mathcal{CS}$) is a finite set of formulas $c_1 : A_1, \ldots, c_n : A_n$ such that c_i is a constant, and F_i an axiom *A0 – A4*. Each derivation in $\mathcal{LP}$ naturally generates the $\mathcal{CS}$ consisting of all formulas introduced in this derivation by the *necessitation* rule.

1.3 Comment. Proof constants in $\mathcal{LP}$ stand for proofs of "simple facts", namely propositional axioms and axioms *A1 – A4*. In a way the proof constants resemble atomic constant terms (*combinators*) of typed combinatory logic (cf. [35]). A constant c_1 specified as $c_1 : (A \rightarrow (B \rightarrow A))$ can be identified with the combinator $\mathbf{k}^{A,B}$ of the type $A \rightarrow (B \rightarrow A)$. A constant c_2 such that $c_2 : [(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))]$ corresponds to the combinator $\mathbf{s}^{A,B,C}$ of the type $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$. The proof variables may be regarded as term variables of combinatory logic, the operation "$\cdot$" as the application of terms. In general an $\mathcal{LP}$-formula $t:F$ can be read as a combinatory term t of the type F. Typed combinatory logic $\mathbf{CL}_{\rightarrow}$ thus corresponds to a fragment of $\mathcal{LP}$ consisting only of formulas of the sort $t:F$ where t contains no operations other than "$\cdot$" and F is a formula built from the propositional letters by "$\rightarrow$" only.

There is no restriction on the choice of a constant c in *R2* within a given derivation. In particular, *R2* allows to introduce a formula $c:A(c)$, or to specify a constant several times as a proof of different axioms from *A0 – A4*. One might restrict $\mathcal{LP}$ to injective constant specifications, i.e. only allowing each constant to serve as a proof of a single axiom $\mathbf{A}$ within a given derivation (although allowing constructions $c\!:\!\mathbf{A}(c)$, as before). Such a restriction would not change the ability of $\mathcal{LP}$ to emulate classical modal logic, or the functional and arithmetical completeness theorems for $\mathcal{LP}$ (below), though it will provoke an excessive renaming of the constants.

For a given constant specification $\mathcal{CS}$ under $\mathcal{LP}(\mathcal{CS})$ we mean $\mathcal{LP}_0$ plus $\mathcal{CS}$. Obviously,

$$F \textit{ is derivable in } \mathcal{LP} \textit{ with a constant specification } \mathcal{CS} \;\Leftrightarrow\; \mathcal{LP}(\mathcal{CS}) \vdash F \;\Leftrightarrow\; \mathcal{LP}_0 \vdash \bigwedge \mathcal{CS} \rightarrow F.$$

1.4 Lemma. (Constructive Necessitation)

$$\vdash F \quad \Rightarrow \; \vdash p\!:\!F \quad \textit{for some proof polynomial } p.$$

1.5 Comment. The differences between deterministic and non-deterministic proof systems are mostly cosmetic. Usual proof systems (Hilbert or Gentzen style) may be considered as deterministic, e.g. a proof derives only the end formula (sequent) of a proof tree. On the other hand, the same systems may be regarded as non-deterministic by assuming that a proof derives all formulas assigned to the nodes of the proof tree.

2 Standard provability interpretation of $\mathcal{LP}$

First order Peano Arithmetic $\mathcal{PA}$ (cf. [7], [8], [24], [33]) is a natural source of proof systems with Gödel numbers of proofs being a natural instrument of internalizing proofs as terms (natural numbers). In principle any system of proofs with a proof checker operation capable of internalizing its own proofs as terms (cf.[32]) can provide a model for $\mathcal{LP}$.

If n is a natural number, then $\overline{n}$ will denote a numeral corresponding to n, i.e. a standard arithmetical term $0'''^{\cdots}$ where $'$ is a successor functional symbol and the number of $'$'s equals n. We will use the simplified notation n for a numeral $\overline{n}$ when it is safe.

2.1 Definition. We assume that $\mathcal{PA}$ contains terms for all primitive recursive functions, called *primitive recursive terms.* Formulas of the form $f(\vec{x}) = 0$ where $f(\vec{x})$ is a primitive recursive term are *standard primitive recursive formulas. A standard* Σ_1 *formula* is a formula $\exists x \varphi(x, \vec{y})$ where $\varphi(x, \vec{y})$ is a standard primitive recursive formula. An arithmetical formula φ is *provably* Σ_1 if it is provably equivalent in $\mathcal{PA}$ to a standard Σ_1 formula; φ is *provably* Δ_1 iff both φ and $\neg\varphi$ are provably Σ_1.

2.2 Definition. A *proof predicate* is a provably Δ_1-formula $Prf(x, y)$ such that for every arithmetical sentence φ

$$\mathcal{PA} \vdash \varphi \quad \Leftrightarrow \quad \text{for some } n \in \omega \quad Prf(n, \ulcorner\varphi\urcorner) \text{ holds}^2.$$

[2] We have omitted bars over numerals for natural numbers $n, \ulcorner\varphi\urcorner$ in the formula *Prf*.

A proof predicate *Prf(x,y)* is *normal* if the following conditions are fulfilled:

1) (*finiteness of proofs*) For every proof k the set $T(k) = \{l \mid Prf(k,l)\}$ is finite. The function from k to the canonical number of $T(k)$ is computable. In particular, this property indicates that the set of theorems proven by k is finite for every k.

2) (*conjoinability of proofs*) For any natural numbers k and l there is a natural number n such that

$$T(k) \cup T(l) \subseteq T(n).$$

2.3 Lemma. *For every normal proof predicate Prf there are computable functions $m(x,y)$, $a(x,y)$, $c(x)$ such that for all arithmetical formulas φ, ψ and all natural numbers k, n the following formulas are valid:*

$$Prf(k, \ulcorner \varphi \rightarrow \psi \urcorner) \wedge Prf(n, \ulcorner \varphi \urcorner) \rightarrow Prf(m(k,n), \ulcorner \psi \urcorner)$$

$$Prf(k, \ulcorner \varphi \urcorner) \rightarrow Prf(a(k,n), \ulcorner \varphi \urcorner), \qquad Prf(n, \ulcorner \varphi \urcorner) \rightarrow Prf(a(k,n), \ulcorner \varphi \urcorner)$$

$$Prf(k, \ulcorner \varphi \urcorner) \rightarrow Prf(c(k), \ulcorner Prf(k, \ulcorner \varphi \urcorner) \urcorner).$$

Note, that the natural arithmetical proof predicate *PROOF(x,y)*

"*x is the code of a derivation* **containing** *a formula with the code y*".

is an example of a normal proof predicate.

2.4 Definition. An arithmetical *interpretation* $*$ of the $\mathcal{LP}$-language has the following parameters:

- a normal proof predicate *Prf* with the functions $m(x,y)$, $a(x,y)$, $c(x)$ as in Lemma 2.4,
- an evaluation of propositional letters by sentences of arithmetic, and
- an evaluation of proof letters and proof constants by natural numbers.

Let $*$ commute with boolean connectives,

$$(t \cdot s)^* = m(t^*, s^*), \quad (t+s)^* = a(t^*, s^*), \quad (!t)^* = c(t^*),$$

$$(t : F)^* = Prf(t^*, \ulcorner F^* \urcorner).$$

A formula $(t:F)^*$ is always provably Δ_1. Note, that $\mathcal{PA}$ (as well as any theory containing certain finite number of arithmetical axioms, e.g. Robinson's arithmetic) is able to derive any true Δ_1 formula, and thus to derive a negation of any false Δ_1 formula (cf. [24]). For a set X of $\mathcal{LP}$-formulas under X^* we mean the set of all F^*'s such that $F \in X$. Given a constant specification $\mathcal{CS}$, an arithmetical interpretation $*$ is a $\mathcal{CS}$-*interpretation* if all formulas from $\mathcal{CS}^*$ are true (equivalently, are provable in $\mathcal{PA}$). An $\mathcal{LP}$-formula F is *valid* (with respect to the arithmetical semantics) if the arithmetical formula F^* is true under all interpretations $*$. F is *$\mathcal{CS}$-valid* if F^* is true under all $\mathcal{CS}$-interpretations $*$.

2.5 Theorem. (Arithmetical completeness of $\mathcal{LP}_0$)

1. *$\mathcal{LP}_0 \vdash F \quad \Leftrightarrow \quad F$ is valid.*
2. *$\mathcal{LP}_0 \vdash F \quad \Leftrightarrow \quad \mathcal{PA} \vdash F^*$ for any interpretation $*$.*

2.6 Corollary. (Arithmetical completeness of $\mathcal{LP}$)

$\mathcal{LP}(\mathcal{CS}) \vdash F \quad \Leftrightarrow \quad F$ is $\mathcal{CS}$-valid.

3 Realization of modal and intuitionistic logics

Let F^o be the result of substituting $\Box X$ for all occurrences of $t{:}X$ in F, and $\Gamma^o = \{F^o \mid F \in \Gamma\}$ for any set Γ of $\mathcal{LP}$-formulas.

3.1 Lemma. *If $\mathcal{LP} \vdash F$, then $\mathcal{S}4 \vdash F^o$.*

Proof. This is a straightforward induction on a derivation in $\mathcal{LP}$.
◀

The goal of the current section is to establish the converse, namely that $\mathcal{LP}$ suffices to realize any $\mathcal{S}4$ theorem. By an *$\mathcal{LP}$-realization* of a modal formula F we mean an assignment of proof polynomials to all occurrences of the modality in F, Let F^r be the image of F under a realization r. Positive and negative occurrences of modality in a formula and a sequent are defined in the usual way.

A realization r is *normal* if all negative occurrences of $\Box$ are realized by proof variables.

3.2 Theorem. *If $\mathcal{S}4 \vdash F$, then $\mathcal{LP} \vdash F^r$ for some normal realization r.*

3.3 Corollary. (Arithmetical completeness of $\mathcal{S}4$.) *$\mathcal{S}4 \vdash F$ iff there is a realization r and a constant specification $\mathcal{CS}$ such that F^r is $\mathcal{CS}$-valid.*

3.4 Comment. It follows from 3.1 and 3.2 that $\mathcal{S}4$ is nothing but a lazy version of $\mathcal{LP}$ when we don't keep track on the proof polynomials assigned to the occurrences of $\Box$. Each theorem of $\mathcal{S}4$ admits a decoding via $\mathcal{LP}$ as a statement about specific proofs.

Let $k(F)$ denote a translation of an intuitionistic formula F into the plain modal language which puts the prefix $\Box$ in front of all atoms and implications in F. It is well-known that $\mathcal{I}nt \vdash F$ *iff* $\mathcal{S}4 \vdash k(F)$ (see, for example, [10]).

3.5 Corollary. (Realization of intuitionistic logic) *For any* $\mathcal{I}nt$*-formula* F

$$\mathcal{I}nt \vdash F \quad \Leftrightarrow \quad \mathcal{LP} \vdash (k(F))^r \text{ for some realization } r.$$

3.6 Corollary. (Arithmetical completeness of $\mathcal{I}nt$.) $\mathcal{I}nt \vdash F$ *iff there is a realization* r *and constant specification* $\mathcal{CS}$ *such that* $k(F)^r$ $\mathcal{CS}$*-valid.*

Kolmogorov's interpretation of intuitionistic logic $\mathcal{I}nt$ as a "*calculus of problems*" ([16]) can be made explicit via $\mathcal{LP}$.

3.7 Definition. Let F be a formula in the intuitionistic propositional language. A formula F is *Kolmogorov realizable* if $\mathcal{LP} \vdash [k(F)]^r$ for some realization r of modalities in $k(F)$ by proof polynomials. *Kolmogorov logic* ($\mathcal{K}$) is the set of all Kolmogorov realizable propositional formulas. Note that the Kolmogorov relizability may be regarded as a direct formalization of the Kolmogorov calculus of problems from [16] under reading "problem solutions" as "proofs".

3.8 Theorem. (Completeness of $\mathcal{I}nt$ with respect to the Kolmogorov realization)

$$\mathcal{I}nt = \mathcal{K}.$$

References

[1] S. Artemov. "Kolmogorov logic of problems and a provability interpretation of intuitionistic logic", *Theoretical Aspects of Reasoning about Knowledge - III Proceedings*, Morgan Kaufman Pbl., pp. 257-272, 1990

[2] S. Artemov and T. Strassen, "Functionality in the Basic Logic of Proofs", *Tech.Rep. IAM 92-004, Department for Computer Science*, University of Bern, Switzerland, 1993.

[3] S. Artemov, "Logic of Proofs", *Annals of Pure and Applied Logic*, v. 67 (1994), pp. 29-59.

[4] S. Artemov, "Operational Modal Logic," *Tech. Rep. MSI 95-29*, Cornell University, December 1995.

[5] S. Artemov, "Proof realizations of typed λ-calculi," *Tech. Rep. MSI 97-2*, Cornell University, May 1997.

[6] J. van Benthem. "Reflections on epistemic logic", *Logique & Analyse*, 133-134, pp. 5-14, 1991

[7] G. Boolos, *The Unprovability of Consistency: An Essay in Modal Logic*, Cambridge University Press, 1979

[8] G. Boolos, *The Logic of Provability*, Cambridge University Press, 1993

[9] S. Buss, "The Modal Logic of Pure Provability", *Notre Dame Journal of Formal Logic*, v. 31, No. 2, 1990

[10] A. Chagrov and M. Zakharyaschev, *Modal Logic*, Oxford Science Publications, 1997.

[11] D. M. Gabbay, *Labelled Deductive Systems*, Oxford University Press, 1994.

[12] K. Gödel, "Eine Interpretation des intuitionistischen Aussagenkalkuls", *Ergebnisse Math. Colloq.*, Bd. 4 (1933), S. 39-40.

[13] K. Gödel, "Vortrag bei Zilsel" (1938), in S. Feferman, ed. *Kurt Gödel Collected Works. Volume III*, Oxford University Press, 1995

[14] R. Goldblatt, "Arithmetical necessity, provability and intuitionistic logic", *Theoria*, 44, pp. 38-46, 1978.

[15] S. Kleene. "On the interpretation of intuitionistic number theory", *Journal of Symbolic Logic*, v. 10, pp. 109-124, 1945

[16] A. Kolmogoroff, "Zur Deutung der intuitionistischen Logik", *Math. Ztschr.*, Bd. 35 (1932), S.58-65.

[17] D. Kozen and J. Tiuryn, "Logic of Programs", in *Handbook of Theoretical Computer Science. Volume B, Formal Models and Semantics*, The MIT Press/Elsevier, pp. 789-840, 1990

[18] S. Kripke, "Semantical considerations on modal logic", *Acta Philosophica Fennica*, 16, pp. 83-94, 1963.

[19] A.V. Kuznetsov and A.Yu. Muravitsky, "The logic of provability", Abstracts of the *4-th All-Union Conference on Mathematical Logic*, p. 73, (Russian), 1976.

[20] E. Lemmon, "New Foundations for Lewis's modal systems", *Journal of Symbolic Logic*, 22, pp. 176-186, 1957.

[21] P. Martin-Löf. "Constructive mathematics and computer programming", in *Logic, Methodology and Philosophy of Science VI*, North-Holland, pp. 153-175, 1982.

[22] P. Martin-Löf. *Intuitionistic Type Theory*, Studies in Proof Theory, Bibliopolis, Naples, 1984.

[23] Yu. Medvedev, “Finite problems”, *Soviet Mathematics Doklady*, v. 3. pp. 227-230, 1962.

[24] E. Mendelson, *Introduction to mathematical logic. Third edition.*, Wadsworth, 1987

[25] G. Mints. “Lewis' systems and system T (1965-1973)”. In *Selected papers in proof theory*, Bibliopolis, Napoli, 1992.

[26] R. Montague. “Syntactical treatments of modality with corollaries on reflection principles and finite axiomatizability”, *Acta Philosophica Fennica*, 16, pp. 153-168, 1963.

[27] J. Myhill, “Some Remarks on the Notion of Proof”, *Journal of Philosophy*, 57, pp. 461-471, 1960

[28] J. Myhill, “Intensional Set Theory”, In: S. Shapiro, ed., *Intensional Mathematics*, North-Holland, pp. 47-61, 1985.

[29] C. Parsons and W. Sieg. “Introductory note to *1938*a*”. In: S. Feferman, ed. *Kurt Gödel Collected Works. Volume III*, Oxford University Press, pp. 62-85, 1995.

[30] S. Shapiro. “Intensional Mathematics and Constructive Mathematics”. In: S. Shapiro, ed., *Intensional Mathematics*, North-Holland, pp. 1-10, 1985.

[31] S. Shapiro. "Epistemic and Intuitionistic Arithmetic". In: S. Shapiro, ed., "Intensional mathematics", North-Holland, pp. 11-46, 1985.

[32] R. Smullyan, *Diagonalization and Self-Reference*, Oxford University Press, 1994

[33] G. Takeuti, *Proof Theory*, North-Holland, 1975

[34] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics. An Introduction*, v. 1, Amsterdam; North Holland, 1988.

[35] A.S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, Cambridge University Press, 1996.

[36] V.A. Uspensky, “Kolmogorov and mathematical logic”, *Sitzungsberichte der Berliner Mathematischen Gessellschaft*, 41-74, Berlin, 1993

[37] V.A. Uspensky and V.E. Plisko “Intuitionistic Logic”, In S.M. Nikol'ski, ed. *A.N. Kolmogorov, Collected works. Matematics and Mechanics*, pp. 394-404, 1985 (in Russian).