Logic of knowledge with justifications

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1 Introduction

How could one describe in logic of knowledge (cf. [4]) the following[1]: an agent receives a product of two very large prime integers. In what sense does the agent know those primes? Another well-known problem: in what sense does an agent know all the derivable formulas regardless to the complexity of their justifications? These and similar problems could be approached in a framework of reasoning about explicit knowledge justifications with atoms $p{:}F$ (*p is a justification for F*) vs. the traditional $\Box F$ (*F is known*). An adequate set of justification terms and operations on them would enable us, in partucular, to evaluate costs of knowledge extraction from a data given. The problem of finding systems for knowledge representation with justifications was discussed by van Benthem in [3]. We wish to think that this paper makes a meaningful step toward to developing such a system.

In this paper we introduce the justification calculus **LPS5** corresponding to the modal logic **S5**. Justification terms in **LPS5** are called extended proof polynomials. They may be regarded as generalization of combinatory terms (therefore typed λ-terms) and naturally subsume proof polynomials for **S4** from [1], [2]. The nonboolean atoms in **LPS5** have form $p : F$ were p is an extended proof polynomial and F is an arbitrary formula, possibly containing other atoms of this kind. The intended semantics of proof polynomials is operations on proofs. However, they admit other natural semantics and may be regarded as epistemic justifications of general kind. Extended proof polynomials disclose the explicit knowledge content of **S5** by realizing all the occurrences of $\Box$'s in any **S5**-derivation. We see the significance of this contribution (compare to the logic of proof for **S4** from [1], [2]) in finding a way of explicit witnessing of negative information, here in the negative introspection principle $\neg\Box B \to \Box\neg\Box B$.

2 Main ideas.

In the explicit counterpart of **S4** the modal principles

$\Box(A \to B) \to (\Box A \to \Box B)$
$\Box A \to \Box\Box A$

are replaced by their dynamic counterparts

[1] We thank Joe Halpern for this example.

$t:(F \to G) \ \to (s:F \to (t\cdot s):G)$
$t:F \ \to\ !t:(t:F)$

respectively, involving functions "$\cdot$" (*application*) and "!" (*proof checker*)[2].

The logic **S5** can reason about negative information concerning knowledge. It is reflected in **S5**-axiom ($\neg\Box B \to \Box\neg\Box B$) which means "*if* B *is not known then this fact is known*". The explicit version of this principle is not very easy to guess since it does not really introduce an operation on proofs but rather a witness of a negative fact about a proof. We suggest two essentially equivalent ways to make **S5** explicit.

Consider an alternative axiomatization of **S5** extending **S4** by the formula D

$$\Box(A \to \neg\Box B) \to (A \to \Box\neg\Box B).$$

It is easy to see that this formulation of **S5** is equivalent to the traditional one above. Indeed, $\mathbf{S4} + D \vdash \neg\Box B \to \Box\neg\Box B$:

$\neg\Box B \to \neg\Box B$
$\Box(\neg\Box B \to \neg\Box B)$
$\Box(\neg\Box B \to \neg\Box B) \to (\neg\Box B \to \Box\neg\Box B)$
$\neg\Box B \to \Box\neg\Box B.$

Likewise, $\mathbf{S5} \vdash D$:

$\Box(A \to \neg\Box B) \to (A \to \neg\Box B)$
$\neg\Box B \to \Box\neg\Box B$
$\Box(A \to \neg\Box B) \to (A \to \Box\neg\Box B)$

The formula D already has a natural explicit counterpart

$$s:(A \to \neg t:B) \to (A \to f(s):(\neg t:B)),$$

where $f(x)$ is a computable operation on proofs. The trick is, of course, in replacing a negative assumption about proofs by a positive one[3]. We have to give some name to this function f: why don't we borrow "?"?

The logic of proofs **LPS5** for **S5** is obtained by adding the new axiom

$$A5: \quad t:(A \to \neg s:B) \to (A \to ?t:(\neg s:B))$$

to the axioms of **LP**.. The traditional axiom $\neg\Box B \to \Box\neg\Box B$ of **S5** can be realized as follows:

$d:(\neg s:B \to \neg s:B)$ – this specifies an axiom constant d as a proof of a tautology
$d:(\neg s:B \to \neg s:B) \to (\neg s:B \to ?d:(\neg s:B))$ – A5
$\neg s:B \to ?d:(\neg s:B)$ – from 1 and 2.

Alternatively, one could build the explicit version of **S5** by introducing a proof constant directly to the main axiom:

$$A5_c: \quad \neg s:B \to c:\neg s:B.$$

[2] One more operation "+" (*choice*) is needed for realization of the whole of **S4**, cf. [1], [2].

[3] There is no need to consider $f(s,t)$ instead of $f(s)$ by two reasons. First, a proof s subsumes all the information about everything it proves, including t. Second, adding $f(s)$ below suffices for realizing the whole of **S5**.

It is easy to see that **LPS5** introduces a new function "?" explicitly whereas in the the above axiom this function is emulated on the metalevel by the series of proof constants. We will show that **LPS5** realizes the whole of **S5** and that **LPS5** is sound and complete with respect to its intended provability interpretation.

3 Definitions and basic lemmas

Definition 3.1 *Proof polynomials* are terms in the language with variables $x_1, \ldots, x_n, \ldots$, constants $c_1, \ldots, c_m, \ldots$ and functional symbols $\cdot$, binary $+$,monadic $!$. *Extended proof polynomials* contain an additional function symbol "?". The formulas of **LPS5** are the ones of the propositional logic[4] with additional atoms of form $p\!:\!F$ where p is an extended proof polynomial and F a formula.

Sometimes we will omit the symbol $\cdot$ in expressions like $(t \cdot s)$ and will skip the parenthesis assuming the following precedences from highest to lowest: $!$, $?$, $\cdot$, $:$, $\neg$, $\wedge$, $\vee$, $\rightarrow$.

Definition 3.2 *An axiom system of* **LPS5** *is the set of the following schemes:*

A0. *Axioms of classical logic in the language of* **LPS5**
A1. $t\!:\!F \rightarrow F$
A2. $t\!:\!(F \rightarrow G) \rightarrow (s\!:\!F \rightarrow (t \cdot s)\!:\!G)$
A3. $s\!:\!F \rightarrow (s+t)\!:\!F$, $t\!:\!F \rightarrow (s+t)\!:\!F$
A4. $t\!:\!F \rightarrow !t\!:\!(t\!:\!F)$
A5. $t\!:\!(F \rightarrow \neg s\!:\!G) \rightarrow (F \rightarrow ?t\!:\!(\neg s\!:\!G))$
R6. *A rule introducing* $\vdash c\!:\!A$*, where* c *is an axiom constant and* A *is axiom* A0 - A5.
R7. *modus ponens.*

Definition 3.3 *A forgetful projection* is a mapping f from **LPS5** formulas to a modal formulas such that $S^f = S$, if S is a proposition variable or constant; f commutes with the boolean connectives; $(t\!:\!F)^f = \Box F^f$.

Lemma 3.4 *If* F *is derivable in* **LPS5** *then* $S5 \vdash F^f$.

Definition 3.5 *A constant specification* associated to a given derivation D is a set of formulas *CS(D)* introduced in D by the rule R6. Let *CS* be a constant specification. We will write **LPS5**$(CS) \vdash A$ if A is derivable in the corresponding logic with constant specifications from *CS* only.

Lemma 3.6 (Constructive necessitation) *If* **LPS5** $\vdash s : A \rightarrow B$ *then there exists an extended proof polynomial* t *such that* **LPS5** $\vdash s\!:\!A \rightarrow t\!:\!B$*. In particular, if* **LPS5** $\vdash B$ *then* **LPS5** $\vdash t\!:\!B$ *for some extended proof polynomial* t.

Proof. By induction on a derivation in **LPS5**. ■

Lemma 3.7 (Realization of S5-axiom) *For any extended proof polynomial* s *and formula* B *one can construct a ground proof polynomial* t *such that* **LPS5** $\vdash \neg s\!:\!B \rightarrow t\!:\!(\neg s\!:\!B)$.

[4]with boolean constants $\top$, $\bot$, proposition letters $S_1, \ldots, S_k, \ldots$, boolean connectives $\wedge, \vee, \rightarrow, \neg$

Proof. Consider the explicit version of the corresponding modal derivation above

1. $c:(\neg s:B \to \neg s:B)$, by R6
2. $c:(\neg s:B \to \neg s:B) \to (\neg s:B \to ?c:(\neg s:B))$, by A5
3. $\neg s:B \to ?c:(\neg s:B)$ from 1 & 2 ■

Remark 3.8 *Note also that all other axioms of* **S5** *have already been realized in* **LPS5** *by A0, A1, A3, A4.*

4 Realization of S5

In order to build a realization of modalities in **S5** derivations by extended proof polynomials we need a Gentzen style formulation of **S5** admitting cut elimination. In this paper we use the system **LS5** from [5].

Definition 4.1 A language of **LS5** contains propositional variables $S_1, \ldots, S_k, \ldots$, boolean constant $\bot$, boolean connective $\to$, modal symbol $\Box$, structure symbols ( ) , < >.

Definition 4.2 System **LS5** has the following *syntactic objects*:

1. *Formula.* A set Fm of **LS5**-formulas is constructed is the usual way. An *atomic formula* is a propositional variable or constant. We will use the capital letters A, B, C, D, F, G (perhaps with indexes) to denote the formulas.

2. *Overline formula.* A set of overline formulas is the set
$$\overline{Fm} := \{\overline{A} \mid A \in Fm\}.$$

3. *Table.* A table is a finite multiset of formulas and overline formulas lied in < >:
$$< A_1, \ldots, A_n, \overline{B_1}, \ldots, \overline{B_m} >.$$
(The empty table <> is also a table). We will use the capital Greek letters $\Gamma, \Delta, \Sigma, \Pi, \Phi$ (perhaps with indexes) to denote the tables.

4. *Sequent.* A sequent is a finite multiset of tables:
$$< \Gamma_1 >, < \Gamma_2 >, \ldots, < \Gamma_k >.$$
We will use the capital letters S, T (perhaps with indexes) to denote the sequents.

Definition 4.3 *A modal translation* of a syntactic object X of system **LS5** is a modal formula X^t constructed the following way:

- $F^t = F$, if F is an **LS5** -formula;
- $(\overline{F})^t = \neg F$ for overline formula $\overline{F}$;
- A modal translation of table $(< A_1, \ldots, \overline{B_m} >)^t = \Box(A_1 \vee\!\!\!\vee \neg B_m)$;
- A modal translation of sequent is a disjunction of the modal translations of tables included in this sequent.

Definition 4.4 *Axioms of* **LS5** are the seqents $< \overline{A}, A >$; $< \overline{\bot} >$. Without loss of generality we may restrict ourselves to the atomic A's only.

Rules of inference of **LS5** is the following rules:

$$\frac{S, < \Sigma >, < \Sigma >}{S, < \Sigma >} \qquad \frac{S, < \Sigma, C, C >}{S, < \Sigma, C >} \qquad (\mathit{contraction})$$

$$\frac{S, < \Sigma >}{S, < \Sigma >, < \Pi >} \qquad \frac{S, < \Sigma >}{S, < \Sigma, C >} \qquad (\mathit{weakening})$$

$$\frac{S, < \Delta, C > \quad T, < \Phi, \overline{C} >}{S, T, < \Delta, \Phi >} \qquad (\mathit{cut})$$

$$\frac{S, < \Delta, \overline{A}, B >}{S, < \Delta, A \to B >} \ (\vdash\to) \qquad\qquad \frac{S, < \Delta, A > \quad S, < \Delta, \overline{B} >}{S, < \Delta, \overline{A \to B} >} \ (\to\vdash)$$

Modal rules:

$$\frac{S, < \Sigma >, < A >}{S, < \Sigma, \Box A >} \ (\vdash \Box) \qquad \frac{S, < \Sigma, \overline{A} >, < \Pi >}{S, < \Sigma >, < \Pi, \overline{\Box A} >} \ (\Box \vdash)$$

Definition 4.5 *A Derivation in* **LS5** is a directed tree with edges labelled by **LS5**-formulas and internal nodes labelled by rules of inference. The leafs of this tree is labelled by axioms of **LS5** and the root of the tree by a *derived sequent.*

Remark 4.6 *The following rules is admissible in* **S5** *:*

$$\frac{S, < \Sigma, \overline{A} >}{S, < \Sigma, \overline{\Box A} >} \ (\Box^-) \quad \mathit{and} \quad \frac{S, < B >}{S, < \Box B >} \ (\Box^+)$$

Example 4.7 *The derivation in $LS5$ of the $S5$-axiom*
$[\neg\Box B \to \Box\neg\Box B]$ *is:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{< B, \overline{B} >}{< B, \overline{B} >, < \bot >}\ (\mathit{weakening})}{< B >, < \overline{\Box B}, \bot >}\ (\Box \vdash)}{< B >, < \Box B \to \bot >}\ (\vdash\to)}{< \Box B >, < \Box B \to \bot >}\ (\Box^+)}{< \Box B, \Box(\Box B \to \bot) >}\ (\vdash \Box) \qquad \dfrac{< \overline{\bot} >}{< \overline{\bot}, \Box(\Box B \to \bot) >}\ (\mathit{weakening})}{\dfrac{< \overline{\Box B \to \bot}, \Box(\Box B \to \bot) >}{< (\Box B \to \bot) \to \Box(\Box B \to \bot) >}\ (\to\vdash)}\ (\vdash\to)$$

Proposition 4.8 ([5]) *a) Let $\Box A$ be a modal formula which is a modal translation of some table $< \Gamma >$. Then*

$$S5 \vdash A \Leftrightarrow LS5 \vdash < \Gamma > .$$

b) The cut rule can be eliminated from any derivation in **LS5**. .

Definition 4.9 Two occurrences of a formula are *related* in a given rule if they appear at the corresponding places in formulas denoted by one letter in the rule: one in the premise and another in the conclusion. In a given derivation we extend the "related" relation by transitivity.

Definition 4.10 Similar occurrences of the symbol $\Box$ in related formulas are called *related.* By a *family* we denote the class of related $\Box$'s.

Definition 4.11 All occurrences of formulas in the derivation has a *positive* or *negative* polarities defined by the usual way:

1. Occurrence of formula F in formula F (as a subformula) has a positive polarity.
2. A corresponding occurrences of subformulas of F in the formula F and in the formula $(G \to F)$, table $< \Gamma, F >$, and sequents with such tables has the same polarities.
3. A corresponding occurrences of subformulas of F in the formula F and in the formula $(F \to G)$ or overline formula $\overline{F}$ has the opposite polarities.

Polarity of $\Box$ is the polarity of the minimal formula that containing this $\Box$.

It is immediate from the definitions thatin a given **LS5**-derivation all related formulas has the same polarities and therefore related $\Box$'s have the same polaraties.

Definition 4.12 We say that the family of $\Box$'s is *positive* if all $\Box$'s in this family have positive polarities. The family is *negative* otherwise.

Definition 4.13 The family is called *essential* if it contains $\Box$ introduced by a rule $(\vdash \Box)$. It is clear that an essential family has a positive polarity.

Definition 4.14 Let X be a syntactic object of the system **LS5** (a formula, an overline formula, a table, or a sequent). By *-realization of* X we call a labelling of occurrences of $\Box$'s and tables $< \Gamma >$ in X by an extended proof polynomials. *Image* X^r of X under realization r is an **LPS5**-formula constructed by induction using the following table:

X	$r(X)$	$(X)^r$
S	S	S
$B \to A$	$r(B) \to r(A)$	$B^r \to A^r$
$\Box A$	$\Box_t A$	$t{:}A$
$\overline{B}$	$\overline{r(B)}$	$\neg B^r$
$< A_1, \ldots, \overline{B_m} >$	$s :< r(A_1), \ldots, r(\overline{B_m}) >$	$s{:}(A_1^r \Cup (\overline{B_m})^r)$
$< \Gamma_1 >, \ldots, < \Gamma_k >$	$r(< \Gamma_1 >), \ldots, r(< \Gamma_k >)$	$(< \Gamma_1 >)^r \Cup (< \Gamma_k >)^r$

Remark 4.15 *Note that for any syntactic object X the forgetful projection of his image under any realization r is a modal translation of X^t:*

$$(X^r)^f = X^t.$$

Definition 4.16 *Realization r(D) of an* **LS5***-derivation D* is the result of applying r to sequents of D.

Definition 4.17 A realization r of an **LS5**-derivation D is *normal* if all negative occurrences of $\Box$ are realized by a proof variables.

Theorem 4.18 (About the realization of $LS5$**)** *For any* **LS5**-*derivation* D *one can construct a normal realization* r *such that for every sequent* T *of* D *the image* $(T)^r$ *of* T *under realization* r *is derivable in* **LP**.

Proof. By cut elimination property of **LS5**, without loss of generality we suppose that D is cut-free and that axioms of D are atomic. We will construct a realization r and an **LP**-derivation D' by the following way:

- Every negative family and every positive non-essential family is realized by a fresh proof variable.
- Pick an essential family f, enumerate all the occurrences of rules $(\vdash \Box)$, which introduce boxes of this family and let k be the total number of this rules for family f. Realize all boxes of the family f by the term $\tilde{u} = (u_1 + u_2 + \ldots + u_k)$, where u_i's are fresh proof variables (called provisional variables).

Provisional variables will be replaced by extended proof polynomials of the "usual" variables during the construction of a realization r.

Derivation D' and realization r will be constructed by induction on D. Initially D' is empty, r is constructed for families like was described above. Note that the realization r is already normal because the negative $\Box$'s are realized by a proof variabless (and will stay like that).

To be able to perform substitutions of terms t_i's for u_i's it is sufficient to check that t_i's are not containing other u_j's. Namely,

1) If the premises's tables are marked by terms not containing u_i's then the conclusion's tables will be marked by terms not containing u_i's too. From this we can conclude that the tables will be marked by the terms containing no u_i's.

2) While substituting the terms t for some u_i's the term t is not containing any u_j's.

We will not discuss these points in each case separately leaving them as routine excersices. For the sake of space we will give here only the clauses concerning modalities.

$(\vdash \Box)$ $\dfrac{\Gamma, t :< \Sigma >, s :< A >}{\Gamma, (p_1 + p_2) :< \Sigma, \Box_{(v_1 + \ldots + v_k)} A >}$;

$(\Sigma)^r = F$, $(A)^r = H$. Let j be a number of this rule in fixed numeration of all the $\Box$'s in the given family introduced by this rule. The $\Box$ just introduced in D has already been labelled by some term $(v_1 + \ldots + v_k)$, were v_i's $(i \neq j)$ is either a provisional variable or an extenden proof polynomial of the usual proof variables and v_j is a provisional variable. Substitute s for all the occurrences of v_j in the derivation. Extend the derivation D' by the following:

0. $(\Gamma)^r \vee t : F \vee s : H$	is already in the derivation
1. $s : H \to \tilde{v} : H$, where $\tilde{v}$ is $v_1 + \ldots + v_{j-1} + s + v_{j+1} \cdots + v_k$	by A3
2. $t : F \to F$	A1
3. $t : F \to (F \vee \tilde{v} : H)$	from 2 in p.c.

4. $t{:}F \rightarrow p_1{:}(F \vee \tilde{v}{:}H)$ — from 3 by Lemma 3.6
5. $s{:}H \rightarrow (F \vee \tilde{v}{:}H)$ — from 1 in p.c.
6. $s{:}H \rightarrow p_2{:}(F \vee \tilde{v}{:}H)$ — from 5 by Lemma 3.6
7. $p_1{:}(F \vee \tilde{v}{:}H) \vee p_2{:}(F \vee \tilde{v}{:}H) \rightarrow (p_1 + p_2){:}(F \vee \tilde{v}{:}H)$ — A5
8. $(\Gamma)^r \vee (p_1 + p_2){:}(F \vee \tilde{v}{:}H)$ — from 0, 4, 6, and 7 in p.c.

The last formula is a realization of conclusion's sequent.

$$(\Box\vdash)\quad \frac{\Gamma, t :< \Sigma, \overline{A} >, s :< \Pi >}{\Gamma, (p_1) :< \Sigma >, (p_2 + p_3) :< \Pi, \overline{\Box_q A} >};$$

$(\Sigma)^r = F$, $(\Pi)^r = G$, $A^r = H$, q is an arbitrary proof variable.

0. $(\Gamma)^r \vee t{:}(F \vee \neg H) \vee s{:}G$ — is already in the derivation
1. $H \wedge (F \vee \neg H) \rightarrow F$ — A0
2. $p{:}H \rightarrow H$ — A1
3. $t{:}(F \vee \neg H) \rightarrow (F \vee \neg H)$ — A1
4. $q{:}H \wedge t{:}(F \vee \neg H) \rightarrow F$ — from 1, 2, and 3 in p.c.
5. $q{:}H \wedge t{:}(F \vee \neg H) \rightarrow p_1{:}F$ — from 4 by Lemma 3.6
6. $\neg q{:}H \rightarrow G \vee \neg q{:}H$ — A0
7. $\neg q{:}H \rightarrow p_2{:}(G \vee \neg q{:}H)$ — from 6 by Lemma 3.6
8. $s{:}G \rightarrow G$ — A1
9. $s{:}G \rightarrow (G \vee \neg q{:}H)$ — from 8 in p.c.
10. $s{:}G \rightarrow p_3{:}(G \vee \neg q{:}H)$ — from 9 by Lemma 3.6
11. $p_2{:}(G \vee \neg q{:}H) \vee p_3{:}(G \vee \neg q{:}H) \rightarrow$ $\rightarrow (p_2 + p_3){:}(G \vee \neg q{:}H)$ — A5
12. $(\Gamma)^r \vee p_1{:}F \vee (p_2 + p_3){:}(G \vee \neg q{:}H)$ — from 0, 5, 7, 10, and 11 in p.c.

■

Corollary 4.19 (About the realization of S5) *For every* **S5** *derivation of a modal formula A one can construct a normal realization r such that* **LPS5** $\vdash A^r$.

5 Provability semantics for LPS5 and S5

Definition 5.1 A *proof predicate* is a provably decidable formula *Prf(x,y)* such that for all φ

$$PA \vdash \varphi \Leftrightarrow Prf(n, \ulcorner\varphi\urcorner) \text{ holds for some } n \in \omega$$

Definition 5.2 A proof predicate *Prf(x,y)* is *normal* if

1. (*finitness of proofs*) For every proof k the set

$$T(k) = \{l \mid Prf(k, l)\}$$

is finite. The function from k to the code of $T(k)$ is computable.

2. (*conjoinability of proofs*) For any proofs k and l there is a proof n such that

$$T(k) \cup T(l) \subseteq T(n)$$

The standard example *Proof(x,y)* of a normal proof predicate is given by the usual linear derivations in the first orfer arithmetic **PA**:

"x is a derivation sequence containing y".

Lemma 5.3 *For each normal proof predicate Prf there are computable functions $x \cdot y$, $x + y$, $!x$ and $?x$ on proofs satisfying identities A2, A3, A4, A5 of* **LPS5**.

Proof. The proof is by example, and the example is the normal proof predicate *Proof(x,y)*.

"$\cdot$" is the usual *application* operation on proofs: $x \cdot y$ is x followed by y and then followed by the list of all formulas B such that $A \leftarrow B$ is in x for some A from y.

"$x + y$" is a concatenation of x and y.

"$!x$" is a proof checking operation that for a given proof x finds a common proof for all the formulas "*Proof(x,A)*" where A is in x. The latter exists, by the provable completeness of **PA** with respect to the provably decidable formulas.

To compute "$?x$" for a given x one has to first recover all formulas of form $A_i \rightarrow \neg Proof(z_i, B_i)$ from x. Then we find proofs c_i such that

$$\neg Proof(z_i, B_i) \rightarrow Proof(c_i, \neg Proof(z_i, B_i)),$$

construct the proofs d_i's of

$$A_i \rightarrow Proof(c_i, \neg Proof(z_i, B_i)),$$

and put $?x$ equal to the concatenation of all d_i's. ■

Definition 5.4 We assume that a normal proof predicate is supplied with some fixed functions for the operations $x \cdot y$, $x + y$, $!x$ and $?x$ on proofs (above). We define *an arithmetical interpretation* $*$ of proof polynomials and **LPS5** formulas as follows:

1. If S is a propositional letter then $S^* = \phi$, where ϕ is an arbitrary arithmetical formula.
2. If x is a proof variable, c a proof constant, then x^*, c^* are arbitrary codes of proofs (natural numbers). The interpretation p^* of each proof polynomial p is then well defined and returns a proof code (a natural number).
3. $*$ commutes with boolean connectives (e.g. $(A \rightarrow B)^* = A^* \rightarrow B^*$).
4. $(t\!:\!F)^* = Prf(t^*, \ulcorner F^* \urcorner)$

Under an interpretation $*$ an **LPS5**-formula F becomes an arithmetical sentence F^*. Note that $(t\!:\!F)^*$ is a substitutional instance of a provably Δ_1 formula.

Theorem 5.5 (The completeness theorem for LPS5)

LPS5 *proves* $F \quad \Leftrightarrow \quad$ *F is valid under any interpretation that respects the constant specification generated by a derivation of F in* **LPS5**.

This is a typical soundness/completeness theorem. Its proof is too long to fit into an abstract. The "soundness" part has de facto been already checked above. The completeness part is the hard one here. However, it goes along the lines of the completeness proof for **LP** from [2]. We first find a cut-free formulation of **LPS5**. Then we consider a formula F which is not derivable in **LPS5** and build a decidable canonical countermodel for F. At the final stage we use a multiple fixed-point equation to simultaneously define an evaluation of proof and propositional letters and a normal proof predicate that constitute an interpretation $*$ falsifying an arithmetical instance F^* of F.

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