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ON FIRST ORDER LOGIC OF PROOFS

SERGEI ARTEMOV AND TATIANA YAVORSKAYA (SIDON)

To the memory of I. G. Petrovskii on the occasion of his 100th anniversary

Abstract. The Logic of Proofs LP solved long standing Gödel's problem concerning his provability calculus (cf. [4]). It also opened new lines of research in proof theory, modal logic, typed programming languages, knowledge representation, etc. The propositional logic of proofs is decidable and admits a complete axiomatization. In this paper we show that the first order logic of proofs is not recursively axiomatizable.

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1. Introduction

The study of provability by means of modal logic was originated by Gödel in 1930s in [11, 12]. He suggested reading the modality $\Box$ as provability; so the formula $\Box F$ is interpreted as *"F is provable"*. This Gödel's proposal led to two substantially different provability interpretations of $\Box F$ each having its own specific mathematical model. We will call them model A and model B.

Model A treats modal sentence $\Box F$ as a formal proposition *"F is derivable in Peano Arithmetic* PA*"*, which in turn can be expressed by an arithmetical formula. Provability Logic consists of modal formulas which are valid under this interpretation. Definitions and detailed presentation of results concerning this approach can be found in [9]. The well known Solovay Completeness Theorem (see [9] or [17]) shows that the propositional Provability Logic is decidable, admits a concise axiomatization and a natural semantical characterization in terms of Kripke models. In fact, Solovay has shown that the modal logic GL[1] axiomatizes all propositional properties of the formal provability predicate. The logic GL is a normal classical modal logic having modal axioms $\Box(P \to Q) \to (\Box P \to \Box Q)$ and $\Box(\Box P \to P) \to \Box P$. The latter is known as Löb's Principle: it is a direct formalization of the well known Löb's theorem [9, 16]. Artemov and Vardanyan in [1, 19, 8] showed that the first

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[1]Also known under the names G, L, K4.W, PRL.

order Provability Logics is not recursively axiomatizable (in fact, the lower complexity bounds for all its reasonable versions are the worst possible). Comprehensive surveys of studies in Provability Logics can be found in [9, 13].

Model B was defined by Gödel axiomatically via his famous modal provability calculus a. k. a. modal logic S4 which eventually led to the Gödel's problem mentioned above[2]. Gödel pointed out in [11, 12] that S4 is incompatible with model A (which reads $\Box F$ as a formal provability assertion *"F is derivable in Peano Arithmetic* PA*"*). More exactly, the reflexivity axiom $\Box F \to F$ of S4 along with the necessitation rule $H \vdash \Box H$ produce a formula $\Box(\Box F \to F)$ which is false under the formal provability interpretation A. Indeed, if F is interpreted as the boolean constant **false**, then $\Box(\Box F \to F)$ asserts the provability of consistency in PA, which does not take place by the Gödel Second Incompleteness Theorem. Despite Gödel's hints in [12] and quite a history of attempts to solve it, the problem of provability semantics for Gödel's provability calculus S4 remained open for more then 60 years until it was solved by the Logic of Proofs LP (cf. [4]) which combined explicit character of λ-calculus with iterative capacities of modal logic. It turned out that Gödel's provability calculus S4 corresponds to the reading of modalities $\Box F$ as explicit provability assertions *"t is a proof of F"* for an appropriate proof term t (called a *proof polynomial*). A complete decidable axiom system of propositional logic of proofs (called the Logic of Proofs LP) was presented in [2, 3, 4]. Logic of Proofs also gives a fair mathematical model for the intended Brouwer–Heyting–Kolmogorov semantics for the propositional intuitionistic logic. In addition, proof polynomials subsume typed λ-calculus and typed combinatory logic. Those features make proof polynomials and the Logic of Proofs attractive for applications in typed programming languages, knowledge representation, automated deduction and verification, etc.

In this paper we consider logic of proofs formulated in the first-order language (see also [7, 15, 20]). In Section 2 we discuss the appropriate first order language of logic of proofs and give exact definitions of arithmetical semantics and of first order logic of proofs. The answers to natural axiomatizability questions for the first order logic of proofs considered in this paper are all negative. In Section 3 we prove that if proofs are represented by special symbols for recursive functions then the corresponding logic is nonaxiomatizable. In Section 4 we consider first order logics with proof constants; we also show that these logics are not axiomatizable.

2. Main definitions

2.1. Some standard definitions and facts. Firstly, let us recall several standard definitions and facts concerning Peano Arithmetic (see [16] for details).

[2]Gödel's problem was raised in [11] where Gödel introduced a special modal calculus for provability (S4) and used it to provide an intended provability interpretation for intuitionistic logic. In this paper Gödel noticed that the straightforward interpretation of S4-modality as of the formal provability operator does not work, thus leaving open the problem of finding a provability interpretation to S4. In [12] Gödel considered this problem once again and specified the format of the intended solution. The Logic of Proofs gives the solution to Gödel's problem in the format suggested in [12].

Hierarchy of arithmetical formulas. By induction on n we define Σ_n-formulas and Π_n-formulas as follows: both Σ_0-formulas and Π_0-formulas are formulas in which all quantifiers are bounded, Σ_{n+1}-formulas have the form $\exists x\varphi$ where φ is Π_n, Π_{n+1}-formulas have the form $\forall x\varphi$ where φ is Σ_n.

A formula is provably Σ_n (or Π_n) if PA proves that it is equivalent to a Σ_n (or, respectively, Π_n) formula. A formula is provably Δ_n if both the formula and its negation are provably Σ_n.

Also we shall use the following definitions and facts from recursion theory (see [14] for details).

A set R of natural numbers is *arithmetical* if there exists an arithmetical formula $\varphi(x)$ such that

$$n \in R \iff \varphi(n) \text{ is true}$$

for every n. *Arithmetical relations* on natural numbers are the ones corresponding to arithmetical sets.

Arithmetical hierarchy. It is well-known that all decidable sets are arithmetical. Arithmetical hierarchy separates arithmetical relations with respect to the least number of alternating quantifiers in a prenex formula which defines the relation in terms of decidable relations. A set is Π_0 if it is decidable. A set is Σ_{n+1} if it is a projection of a Π_n-set. A set is Π_n if its complement is a Σ_n-set. For example, the class Σ_1 consists of all recursively enumerable sets.

This definition can be easily modified for sets of words in any finite alphabet with the help of the appropriate coding of words by numbers (i. e., Gödel numbering). So we can speak about arithmetical sets of formulas and so on.

In what follows X and Y are sets of natural numbers. A set X is called *m-reducible* to Y if there exists a total recursive function f such that for every n

$$n \in X \iff f(n) \in Y.$$

If this function f is one-to-one then we say that X is *1-reducible* to Y.

Classes Σ_n and Π_n are close under m-reducibility: if $Y \in \Sigma_n$ (Π_n) and X is m-reducible to Y then $X \in \Sigma_n$ (Π_n). In particular, if Y is decidable or recursively enumerable then so is X.

Let K be any of classes Σ_n and Π_n. We say that a set X is *K-hard* if any set from K is 1-reducible to X. If X is K-hard and $X \in K$ then X is called *K-complete.*

2.2. The language of first order logic of proofs. By predicate language $\mathcal{L}$ we mean the first order language without function symbols and equality containing countable set of predicate letters of any arity.

First order provability logic is formulated in the extension of the language $\mathcal{L}$ by the modal operator $\Box$ for provability; the modal formula $\Box F$ is interpreted as a proposition about provability *"there exists a proof of F"*. As we mentioned before, the transition from Provability Logic to the Logic of Proofs, generally speaking, consists in eliminating the existential quantifiers hidden in the modality of provability and replacing them by the concrete proofs. In the propositional logic the formula F above represents a proposition without parameters. So the proof of F needs not depend on the parameters, thus it can be represented by *proof variables* ranging over codes of proofs, or natural numbers.

In order to define an appropriate language of quantified logic of proofs let us look at the provability formula $\Box A(x)$, where a formula A under the modality depends on a parameter x. It represents the proposition *"for a given x the formula $A(x)$ is provable"* which contains x as a parameter (see [1, 19]). If we write down the existential quantifier on proofs hidden in the provability operator, then we obtain the proposition *"for a given x there exists y such that y is a code of proof of $A(x)$"* with two parameters x and y. Then the standard procedure of skolemization (which consists, roughly, in replacing existential quantifiers by function symbols) provides us with a function f which produces a proof of $A(x)$ being given a number x.

$\Box A(x)$ $\Uparrow$ intended interpretation $\Downarrow$ — *"for a given x there exists y which is a proof of $A(x)$"*

$⟦f(x)⟧A(x)$ $\Uparrow$ intended interpretation $\Downarrow$ — *"for a given x, $f(x)$ is a proof of $A(x)$"*

"for a given x there exists y which is a proof of $A(x)$" $\Leftarrow$ skolemization $\Rightarrow$ *"for a given x, $f(x)$ is a proof of $A(x)$"*

The language $\mathcal{L}^f$ described below captures Skolem functions of this sort. It is the extension of the predicate language $\mathcal{L}$ by symbols for recursive functions on proofs and operational symbol $⟦\cdot⟧(\cdot)$ for proof predicate.

Definition 1. The *language* $\mathcal{L}^f$ contains:

- individual variables x, y, z, ...,
- countable set of predicate letters of any arity P, Q, R, ...,
- countable set of proof functional letters of any arity: f, g, h, ...,
- operational symbol $⟦\cdot⟧(\cdot)$,
- boolean connectives and quantifiers.

Let $\mathcal{L}^c$ denote the fragment of the language $\mathcal{L}^f$ which contains only constants on proofs (i. e., proof functional letters of arity 0).

Formulas of the language $\mathcal{L}^f$ are defined in the standard way with the only additional clause for formulas representing proof predicate. We denote the set of formulas by $Fm(\mathcal{L}^f)$. So:

- $\bot$ and $P(x_1, \ldots, x_n)$ are *atomic formulas*, where P is a predicate letter and x_i are individual variables;
- the set of formulas is closed under the boolean connectives and quantifiers;
- if F is a formula, g^n is a proof functional letter of arity n and $x_1, \ldots, x_n$ are individual variables, then $⟦g^n(x_1, \ldots, x_n)⟧F$ is a formula.

The set of free variable of a formula F is denoted by *Free Var*(F), where

$$\mathit{Free\,Var}(⟦g^n(x_1, \ldots, x_n)⟧F) = \mathit{Free\,Var}(F) \cup \{x_1, \ldots, x_n\}.$$

Remark 1. Note that proof functional letters of the language $\mathcal{L}^f$ are not function symbols of a usual first order language in the standard meaning of the term. They can appear in a formula only in the scope of $⟦\cdot⟧$-part of the operational symbol $⟦\cdot⟧(\cdot)$.

2.3. Arithmetical interpretation. Let us describe the intended arithmetical semantics for $\mathcal{L}^f$. In what follows PA denotes Peano Arithmetic , TA stands for "Truth Arithmetic", that is, the set of arithmetical formulas which are true in the standard model (natural numbers).

In order to represent recursive functions we consider Peano Arithmetic enriched by recursive ι-terms. For every formula $\varphi(x_1, \ldots, x_n, y)$ such that

$$\mathsf{PA} \vdash \varphi(x_1, \ldots, x_n, y) \wedge \varphi(x_1, \ldots, x_n, z) \rightarrow y = z,$$

the expression $\iota y.\varphi(x_1, \ldots, x_n, y)$ is a *ι-term* of arity n. It is supposed to represent a (partial) function $f(x_1, \ldots, x_n)$ which assigns to $x_1, \ldots, x_n$ the unique y such that $\varphi(x_1, \ldots, x_n, y)$. We use the expression $\psi(\iota y.\varphi(\vec{x}, y))$ to abbreviate the formula $\exists y(\psi(y) \wedge \varphi(\vec{x}, y))$.

Definition 2. A ι-term $\iota y.\varphi(\vec{x}, y)$ is called *recursive* if the formula φ is provably Σ_1 in PA. If $\mathsf{PA} \vdash \forall \vec{x}\, \exists y\ \varphi(\vec{x}, y)$, then the corresponding ι-term is called *provably total* (the corresponding function represented by provably total ι-term is also called *provably total*).

The following lemma says that all recursive functions can be represented as recursive ι-terms; it is a reformulation of the theorem on arithmetical representation of recursive functions.

Lemma 1. *The following holds*:

1. *Every recursive ι-term represents a recursive function. And vice versa, every recursive function can be represented by a recursive ι-term.*
2. *Every primitive recursive function can be represented by a provably total recursive ι-term.*

Example 1. By $\ulcorner\varphi(\dot{y}_1, \ldots, \dot{y}_n)\urcorner$ we denote provably total recursive ι-term for primitive recursive function $\lambda k_1, \ldots, k_n.\ulcorner\varphi(k_1, \ldots, k_n)\urcorner$ that being given any k_1, $\ldots$, k_n calculates the Gödel number of a formula $\varphi(k_1, \ldots, k_n)$.

Definition 3. Let T be a recursively enumerable arithmetical theory, $\mathsf{PA} \subseteq T \subset \mathsf{TA}$. A *proof predicate for a theory T* is a provably Δ_1-formula $\mathit{Prf}(x, y)$ which enumerates theorems of T in PA, that is for any arithmetical sentence φ

$$T \vdash \varphi \iff \mathsf{PA} \vdash \mathit{Prf}(n, \ulcorner\varphi\urcorner) \quad \text{for some } n,$$

where $\ulcorner\varphi\urcorner$ stands for the Gödel number of a formula φ.

Example 2. Here are some examples of proof predicates. The standard Gödel proof predicate for T is an arithmetical formula

$$\mathit{Proof}_T(x, y) \rightleftharpoons \text{“}x \text{ is a Gödel number of a derivation in } T \text{ and } y \text{ is the Gödel number of the last formula in it”.}$$

One can consider the multi-conclusion version of this predicate:

$$\mathit{PROOF}_T(x, y) \rightleftharpoons \text{“}x \text{ is a Gödel number of a derivation in } T \text{ and } y \text{ is the Gödel number of some formula in it”.}$$

Here is another version of multi-conclusion proof predicate for T:

$$\mathit{Prf}_T(x, y) \rightleftharpoons \text{“}x \textit{ is a Gödel number of a finite set of derivations in } T \textit{ and } y \textit{ is the Gödel number of the last formulas in one of them”}.$$

Now we are ready to define an arithmetical interpretation of the language $\mathcal{L}^f$.

Definition 4. An arithmetical interpretation $* = (\mathit{Prf}, \varepsilon)$ of the language $\mathcal{L}^f$ has the following parameters:

- a proof predicate Prf for a recursively enumerable theory $\mathsf{PA} \subseteq T \subset \mathsf{TA}$;
- an evaluation ε, which assigns to proof functional letters provably recursive arithmetical ι-terms of the same arity, and maps atomic formulas to arithmetical formulas with the same free variables. We assume that ε commutes with renaming of free variables.

Given an arithmetical interpretation $*$ one could translate all formulas of the language $\mathcal{L}^f$ by arithmetical formulas in the following canonical way. For atomic formulas $Q^* \rightleftharpoons \varepsilon Q$, $*$ commutes with boolean connectives and quantifiers, and

$$([\![g(\vec{x})]\!]A(\vec{y}))^* \rightleftharpoons \mathit{Prf}(\varepsilon g(\vec{x}), \ulcorner A^*(\dot{\vec{y}})\urcorner).$$

2.4. First order logics of proofs. Let us give the definition of first order logic of proofs. In what follows U stands for an arithmetical theory such that $\mathsf{PA} \subseteq U \subseteq \mathsf{TA}$.

Definition 5. Suppose that Prf is a proof predicate for a recursively enumerable theory T, where $\mathsf{PA} \subseteq T \subset \mathsf{TA}$. We define two versions of the *logic of proofs for the predicate Prf*:

$$\mathcal{QLP}^f_{\mathit{Prf}}(U) \rightleftharpoons \{A \in \mathit{Fm}(\mathcal{L}^f) \colon \text{for any interpretation } * = (\mathit{Prf}, \epsilon)\ U \vdash A^*\},$$

$$\mathcal{QLP}^c_{\mathit{Prf}}(U) \rightleftharpoons \{A \in \mathit{Fm}(\mathcal{L}^c) \colon \text{for any interpretation } * = (\mathit{Prf}, \epsilon)\ U \vdash A^*\}.$$

For a given class of proof predicates $\mathcal{K}$ we define the corresponding logics of proofs by

$$\mathcal{QLP}^f_{\mathcal{K}}(U) \rightleftharpoons \bigcap_{\mathit{Prf} \in \mathcal{K}} \mathcal{QLP}^f_{\mathit{Prf}}(U),$$

$$\mathcal{QLP}^c_{\mathcal{K}}(U) \rightleftharpoons \bigcap_{\mathit{Prf} \in \mathcal{K}} \mathcal{QLP}^c_{\mathit{Prf}}(U).$$

The logics $\mathcal{QLP}^f_{\mathcal{K}}(U)$ and $\mathcal{QLP}^c_{\mathcal{K}}(U)$ describe all universal properties of the predicates $\mathit{Prf} \in \mathcal{K}$ that can be proved in U. We shall be interested mostly in the cases $U = \mathsf{PA}$ and $U = \mathsf{TA}$.

Remark 2. Propositional logics of proofs for PA and TA coincide (cf. [6]). However, this property does not hold in the predicate case even for the language with proof constants. In order to show that $\mathcal{QLP}^c(\mathsf{PA}) \neq \mathcal{QLP}^c(\mathsf{TA})$ we consider weak reflexivity principle $[\![t]\!]P(x) \to P(x)$. Obviously it is arithmetically valid and thus belongs to $\mathcal{QLP}^c(\mathsf{TA})$. But this principle cannot be proven in PA under the interpretation $* = (\mathit{Prf}, \epsilon)$, where $\epsilon P \rightleftharpoons \neg \mathit{Proof}_{\mathsf{PA}}(x, \ulcorner\bot\urcorner)$, $\epsilon t = 1$ and proof predicate $\mathit{Prf}(z, y)$ is defined by the formula

$$\mathit{Proof}_{\mathsf{PA}}(z, y) \vee (z = 1 \wedge \exists x < y\ (y = \ulcorner \neg \mathit{Proof}_{\mathsf{PA}}(\dot{x}, \ulcorner\bot\urcorner)\urcorner)).$$

Actually, from the definition of Prf we immediately conclude that

$$\mathsf{PA} \vdash Prf(1, \ulcorner P^*(\dot{x})\urcorner).$$

Gödel's second incompleteness theorem provides that $\mathsf{PA} \nvdash P^*(x)$, a contradiction.

3. Lower complexity bounds for $\mathcal{QLP}^f_{\mathcal{K}}(U)$

In this section we show that the logic $\mathcal{QLP}^f_{\mathcal{K}}(\mathsf{TA})$ is nonarithmetical and that $\mathcal{QLP}^f_{\mathcal{K}}(\mathsf{PA})$ is Π_2-hard for any class of proof predicates $\mathcal{K}$. We obtain these facts as a corollary of a more general theorem which gives the lower complexity bounds for all the logics $\mathcal{QLP}^f_{\mathcal{K}}(U)$.

3.1. Preliminaries.

Definition 6. Arithmetical formula $\varphi(x)$ is *decidable in a theory* U if for any natural number $n \in \omega$ either $U \vdash \varphi(n)$ or $U \vdash \neg\varphi(n)$. If φ is decidable in U, then the set of form $\{n\colon U \vdash \varphi(n)\}$ is called *decidable in* U. A set belongs to the class Π_n in U (Σ_n in U) if it is of complexity Π_n (Σ_n) with respect to a set decidable in U.

Lemma 2 [19, 9]**.** *Let the set P be Π_2 in U. Then there exists a formula $Q(y, z)$ decidable in U such that*

$$n \in P \iff \forall x\ \exists y > x\ U \vdash Q(y, n).$$

We also need the following lemma which is a strengthened version of the well-known Tennenbaum theorem. The theorem says that any model of Peano Arithmetic in which addition and multiplication are recursive functions is isomorphic to the standard model. The lemma says that the same fact holds for a finite fragment of Peano Arithmetic.

Lemma 3 [1]**.** *There exists a finite set Ten of theorems of Peano Arithmetic such that every model of Ten with the domain ω in which $+$ (addition) and $\times$ (multiplication) are recursive functions is isomorphic to the standard model.*

3.2. The main theorem.

Theorem 1. *Suppose that U is any arithmetically correct theory and $\mathcal{K}$ is an arbitrary nonempty class of proof predicates. Then any set, which is Π_2 in U, is m-reducible to $\mathcal{QLP}^f_{\mathcal{K}}(U)$.*

The proof of this theorem goes on the lines of proofs of similar facts for predicate provability logics (cf. [1, 19, 8]).

Proof. Let P be Π_2 in U. Lemma 2 provides us with a formula $Q(y, z)$ decidable in U such that for any n

$$n \in P \iff \forall x\ \exists y > x\ U \vdash Q(y, n).$$

Let us describe an algorithm that performs the reduction of P to $\mathcal{QLP}^f_{\mathcal{K}}(U)$.

Description of the reduction algorithm. Consider the predicate language EAM consisting of a binary predicate symbol E and two ternary predicate symbols A and M. We apply the standard procedure of replacing function symbols by predicate symbols to arithmetical language, where E, A, and M stand for equality, addition, and multiplication predicates respectively. Let $\{\varphi\}$ denote the result of the described translation of an arithmetical formula φ into the language EAM.

Let $\mathcal{T}$ denote the conjunction of all formulas $\{\varphi\}$ ($\varphi \in \mathit{Ten}$) and standard axioms in the language EAM expressing basic properties of equality for the predicate E and functionality of the predicates A and M. Put

$$Eq \rightleftharpoons \forall x,\, y\ [E(x,\, y) \to (W(x) \leftrightarrow W(y))].$$

Suppose that $W(x)$ is a unary predicate symbol and $p(x,\, y,\, z)$, $q(x,\, y,\, z)$, $r(x,\, y)$, $t(x)$ are proof functional letters of the indicated arity. We define a formula that expresses decidability of E, A, M, and W in the following way:

$$\begin{aligned} D \rightleftharpoons \forall x,\, y,\, z\ [\,&(A(x,\, y,\, z) \leftrightarrow [\![p(x,\, y,\, z)]\!]A(x,\, y,\, z)) \\ &\wedge (M(x,\, y,\, z) \leftrightarrow [\![q(x,\, y,\, z)]\!]M(x,\, y,\, z)) \\ &\wedge (E(x,\, y) \leftrightarrow [\![r(x,\, y)]\!]E(x,\, y)) \\ &\wedge (W(x) \leftrightarrow [\![t(x)]\!]W(x))\,]. \end{aligned}$$

Let $S(x,\, y)$ be a natural arithmetical Σ_1-formula expressing the relation *"Turing machine having number x terminates on input y"*. Now we can describe the desired algorithm. For any $n \in \omega$ it produces the formula

$$\Phi_n \rightleftharpoons \mathcal{T} \wedge D \wedge Eq \to \exists x\ \exists y\ (x\{<\}y \wedge \{Q\}(y,\, n)) \wedge \forall z\ (W(z) \leftrightarrow \{S\}(x,\, z)). \quad (1)$$

Let us show that a (recursive) function $n \mapsto \Phi_n$ performs the reduction of P to $\mathcal{QLP}^f_{\mathcal{K}}(U)$. It suffices to establish that

$$\forall x\ \exists y > x\ U \vdash Q(y,\, n) \iff \forall \mathit{Prf} \in \mathcal{K}\ \forall * = (\mathit{Prf},\, \varepsilon)\ U \vdash \Phi_n^*. \quad (2)$$

Proof of ($\Rightarrow$). Suppose $\forall x\ \exists y > x\ U \vdash Q(y,\, n)$. Let Prf be an arbitrary proof predicate from $\mathcal{K}$ and ε an arbitrary evaluation. Consider the interpretation $* = (\mathit{Prf},\, \varepsilon)$. Let us prove that $U \vdash \Phi_n^*$.

Step 1. Since Prf is provably decidable and since arithmetical terms assigned to functional variables are provably total, we conclude that for any evaluation ε

$$\mathsf{PA} \vdash D^* \to \text{“}\varepsilon E,\, \varepsilon A,\, \varepsilon M, \text{ and } \varepsilon W \text{ are decidable”}. \quad (3)$$

For example, the decision algorithm for εA for given x, y, and z calculates the value of $\varepsilon p(x,\, y,\, z)$ and then checks whether $\mathit{Prf}(\varepsilon p(x,\, y,\, z),\, \ulcorner \varepsilon A(\dot{x},\, \dot{y},\, \dot{z})\urcorner)$ holds. The formula D^* guarantees that $\varepsilon A(x,\, y,\, z)$ is true if the answer is positive and false otherwise. Decision algorithms for the remaining predicates work similarly.

Step 2. Arithmetical formula $R(x,\, y)$ defined below expresses the relation *"y represents a number x in the model defined by the arithmetical interpretation $*$"*

$$\begin{aligned} R(x,\, y) \rightleftharpoons\ &\text{“there exists a finite sequence } s \text{ of length } x+1, \\ &\text{such that } (s)_0 = 0^*,\ (s)_x = y, \text{ and } \forall z < x\ A^*(1^*,\, (s)_z,\, (s)_{z+1})\text{”}, \end{aligned}$$

where constants 0^* and 1^* are defined in terms of A^* and M^* in the standard manner:

$$(x = 0^*) \rightleftharpoons A^*(x, x, x), \qquad (x = 1^*) \rightleftharpoons M^*(x, x, x) \wedge \neg A^*(x, x, x).$$

The following properties of $R(x, y)$ are established in [19, 9].

(R1) $U \vdash \mathcal{T}^* \wedge R(z, z_1) \wedge R(z, z_2) \to E^*(z_1, z_2)$;
(R2) $U \vdash \mathcal{T}^* \to \forall a\, \exists b\, R(a, b)$;
(R3) (formalized Tennenbaum theorem)

$$U \vdash \mathcal{T}^* \wedge \text{“}\varepsilon E,\ \varepsilon A, \text{ and } \varepsilon M \text{ are decidable”} \to \forall y\, \exists x\, R(x, y).$$

Step 3. We can show by induction on formula $\varphi(\vec{x})$ that

$$U \vdash \mathcal{T}^* \wedge \forall b\, \exists a\, R(a, b) \wedge R(\vec{x}, \vec{y}) \to (\varphi(\vec{x}) \leftrightarrow \{\varphi\}^*(\vec{y})),$$

where $\vec{x} = (x_1, \ldots, x_m)$ denotes the set of all free variables of a formula φ and $R(\vec{x}, \vec{y})$ is an abbreviation for $\bigwedge_{i=1}^m R(x_i, y_i)$. Hence from (3) and (R3) it follows that

$$U \vdash \mathcal{T}^* \wedge D^* \wedge R(\vec{x}, \vec{y}) \to (\varphi(\vec{x}) \leftrightarrow \{\varphi\}^*(\vec{y})). \tag{4}$$

Step 4. There exists a natural number k such that

$$U \vdash D^* \to [\exists z\, (R(v, z) \wedge W^*(z)) \leftrightarrow S(k, v)]. \tag{5}$$

Actually, according to (3) from D^* it follows that the relations A^*, M^* and E^* are decidable. In view of the definition, $R(v, z)$ is recursively enumerable. Since relation $W^*(z)$ is recursive by (3), the set $\{v\colon \exists z\, (R(v, z) \wedge W^*(z))\}$ is enumerable too. This provides us with the desired k.

Step 5. Let us show that

$$U \vdash \mathcal{T}^* \wedge D^* \wedge Eq^* \wedge R(v, z) \to (W^*(z) \leftrightarrow S(k, v)). \tag{6}$$

We reason in U. In view of $\mathcal{T}^* \wedge D^* \wedge Eq^*$ and (5), from $W^*(z)$ we immediately obtain $S(k, v)$. For the converse assume that $S(k, v)$. From (5) we get $\exists z_1\, (R(v, z_1) \wedge W^*(z_1))$. According to (R1), $R(v, z)$ and $R(v, z_1)$ imply $E^*(z, z_1)$. In view of Eq^*, we conclude that $W^*(z_1) \leftrightarrow W^*(z)$, whence $W^*(z)$.

Step 6. According to our original assumption (see (2)), there exists a number l such that $k < l$ and $U \vdash Q(l, n)$. Then $U \vdash k < l \wedge Q(l, n)$. Using (4) we derive

$$U \vdash \mathcal{T}^* \wedge D^* \wedge R(k, x) \wedge R(l, y) \to x\{<\}y \wedge \{Q\}^*(y, n). \tag{7}$$

Step 7. Reason in U. From (6) and (4) it follows that

$$\mathcal{T}^* \wedge D^* \wedge Eq^* \wedge R(k, x) \wedge R(v, z) \to (W^*(z) \leftrightarrow \{S\}^*(x, z)).$$

Applying (R3), (7), (R2) and doing standard manipulations in predicate calculus we conclude the desired

$$\mathcal{T}^* \wedge D^* \wedge Eq^* \to \exists x\, \exists y\, [x\{<\}y \wedge \{Q\}^*(y, n) \wedge \forall z\, (W^*(z) \leftrightarrow \{S\}^*(x, z))]. \quad \square$$

Proof of ($\Leftarrow$). Suppose that $U \vdash \Phi_n^*$ under every arithmetical interpretation $* = (\mathit{Prf}, \varepsilon)$, where $\mathit{Prf} \in \mathcal{K}$. Let us show that $\forall m\, \exists l > m\; U \vdash Q(m, n)$.

Step 1. Let ϵ denote the standard arithmetical evaluation of the language EAM which assigns the equality, addition, and multiplication predicates to the predicate letters E, A, and M respectively:

$$\begin{aligned} \varepsilon E &\rightleftharpoons (x = y), \\ \varepsilon A &\rightleftharpoons (x + y = z), \\ \varepsilon M &\rightleftharpoons (xy = z). \end{aligned} \tag{8}$$

Step 2. We fix a proof predicate $Prf \in \mathcal{K}$ and an arbitrary $m \in \omega$ and consider the evaluations ε_k $(k = 1, \ldots, m)$ which are extensions of ε defined as follows:

$$\begin{aligned} \varepsilon_k W &\rightleftharpoons (z = k), \\ \varepsilon_k r &\rightleftharpoons \mu w.Prf(w, \ulcorner \dot{x} = \dot{y} \urcorner) \wedge (x = y), \\ \varepsilon_k p &\rightleftharpoons \mu w.Prf(w, \ulcorner \dot{x} + \dot{y} = \dot{z} \urcorner) \wedge (x + y = z), \\ \varepsilon_k q &\rightleftharpoons \mu w.Prf(w, \ulcorner \dot{x}\dot{y} = \dot{z} \urcorner) \wedge (xy = z), \\ \varepsilon_k t &\rightleftharpoons \mu w.Prf(w, \ulcorner k = k \urcorner) \wedge (z = k). \end{aligned} \tag{9}$$

For any $k = 0, \ldots, m$ consider the interpretation $*_k = (Prf, \varepsilon_k)$.

Step 3. It can be easily seen that $U \vdash \mathcal{T}^{*_k} \wedge D^{*_k} \wedge Eq^{*_k}$. Since $U \vdash \Phi_n^{*_k}$ for all $k = 0, \ldots, m$, we have

$$U \vdash \exists x\, \exists y\, [x < y \wedge Q(y, n) \wedge \forall z\, ((z = k) \leftrightarrow S(x, z))].$$

Therefore

$$U \vdash \exists x\, \exists y\, [x < y \wedge Q(y, n) \wedge \bigwedge_{i=0}^{m} ((i = k) \leftrightarrow S(x, i))]. \tag{10}$$

Step 4. For any $k = 0, \ldots, m$ consider the number x_k satisfying (10). Let us show that $x_k > m$ for some k. Suppose that $x_k < m$ for all $k = 0, \ldots, m$. Applying the pigeonhole principle, we obtain $x_{k_1} = x_{k_2}$ for some $k_1 \neq k_2$. For interpretations $*_{k_1}$ and $*_{k_2}$ consider the conjunct corresponding to $i = k_1$ in formula (10). We obtain respectively that

$$U \vdash (k_1 = k_1) \to S(x_{k_1}, k_1) \quad \text{and} \quad (k_1 = k_2) \leftrightarrow S(x_{k_2}, k_1) \text{ is true.}$$

From $U \vdash k_1 = k_1$ it follows that $U \vdash S(x_{k_1}, k_1)$. Since $x_{k_1} = x_{k_2}$ we can derive $U \vdash S(x_{k_2}, k_1)$. Therefore formula $k_1 = k_2$ with $k_1 \neq k_2$ is true in the standard model.

The contradiction obtained shows that $x_k \geq m$ for some $k = 0, \ldots, m$. In accordance with (10), there exists a natural number $l > x_k$ such that $Q(l, n)$ holds. Since Q is decidable in U we conclude that $U \vdash Q(l, n)$. Then we have $l > m$ and $U \vdash Q(l, n)$. □

3.3. Corollaries.

Corollary 1. *Suppose that Prf is a proof predicate. Then any set that is* Π_2 *in* U *is m-reducible to* $\mathcal{QLP}^f_{Prf}(U)$.

Corollary 2. *For every proof predicate Prf the set* $\mathcal{QLP}^f_{Prf}(\mathsf{TA})$ *is not arithmetical.*

Proof. Note that all arithmetical sets are decidable in TA. According to Theorem 1, all these sets can be reduced to $\mathcal{QLP}^f_{Prf}(\mathsf{TA})$. Thus $\mathcal{QLP}^f_{Prf}(\mathsf{TA})$ is nonarithmetical. □

Remark 3. It is immediate from the definition that the logic $\mathcal{QLP}^f_{Prf}(\mathsf{TA})$ belongs to the complexity class $\Pi^0_1(\mathsf{TA})$. According to Corollary 2, this logic is nonarithmetical. This result can be strengthened using a method from [8]. Namely, it can be shown that $\mathcal{QLP}^f_{Prf}(\mathsf{TA})$ is $\Pi^0_1(\mathsf{TA})$-complete.

Corollary 3. *For each proof predicate Prf the set* $\mathcal{QLP}^f_{Prf}(\mathsf{PA})$ *is* Π_2*-complete.*

Proof. From the definitions, one can easily see that $\mathcal{QLP}^f_{Prf}(\mathsf{PA})$ belongs to Π_2. On the other hand, all recursive relations are decidable in PA, whence, by Theorem 1, any Π_2-set can be reduced to $\mathcal{QLP}^f_{Prf}(\mathsf{PA})$. □

4. First order logics with constants on proofs

In this section we find lower complexity bounds for first order logics of proofs formulated in the language with constants on proofs $\mathcal{L}^c$. We consider two cases:

(1) the class of all proof predicates and the class of all proof predicates for a given recursively enumerable theory $\mathsf{PA} \subseteq T \subseteq \mathsf{TA}$;
(2) any class consisting of normal multi-conclusion proof predicates, that is, proof predicates which imitate real proof processes (for the precise definition see page 486);

4.1. Logic of all proof predicates. The logic of all proof predicates formulated in the language $\mathcal{L}^c$ is denoted by $\mathcal{QLP}^c(U)$. Let T be any recursively enumerable arithmetical theory, $\mathsf{PA} \subseteq T \subset \mathsf{TA}$. The logic of all proof predicates for T is denoted by $\mathcal{QLP}^c_T(U)$.

Theorem 2. *For every arithmetical theory* U *such that* $\mathsf{PA} \subseteq U \subseteq \mathsf{TA}$*,*

1) *any set which is* Π_2 *in* U *is* m*-reducible to* $\mathcal{QLP}^c(U)$*;*
2) *any set which is* Π_2 *in* U *is* m*-reducible to* $\mathcal{QLP}^c_T(U)$*.*

Proof. The proof of both 1) and 2) is a slight modification of the proof of Theorem 1. We describe changes needed for 2). Let P be any set which is Π_2 in U. The algorithm performing a reduction of P to $\mathcal{QLP}^c_T(U)$ to every natural number $n \in \omega$ assigns the following formula Φ_n defined similarly to (1)

$$\Phi_n \rightleftharpoons \mathcal{T} \wedge D \wedge Eq \to \exists x\, \exists y\, (x\{<\}y \wedge \{Q\}(y, n) \wedge \forall z\, (W(z) \leftrightarrow \{S\}(x, z))$$

with the only difference in formula D in which we have to replace proof functional symbols by proof constants. Now D is defined as follows

$$\begin{aligned} D \rightleftharpoons \forall x, y, z\, [& (A(x, y, z) \leftrightarrow [\![p]\!]A(x, y, z)) \\ & \wedge (M(x, y, z) \leftrightarrow [\![q]\!]M(x, y, z)) \\ & \wedge (E(x, y) \leftrightarrow [\![r]\!]E(x, y)) \\ & \wedge (W(x) \leftrightarrow [\![t]\!]W(x))], \end{aligned}$$

where p, q, r, and t are proof constants.

It remains to show that recursive function $n \mapsto \Phi_n$ performs a reduction of P to $\mathcal{QLP}^c_T(U)$, that is

$$\forall x\ \exists y > x\ U \vdash Q(y, n) \iff \forall * = (Prf_T, \varepsilon)\ U \vdash \Phi_n^*. \tag{11}$$

A detailed analysis of the proof of Theorem 1 shows that the proof of the left-to-right implication does not change. To establish the converse, in the proof of Theorem 1 we considered an interpretation of a special form (see (8) and (9)). Here we do the same thing, but we need to define a specific proof predicate on which this interpretation is based. An appropriate proof predicate can be defined by the formula

$$\begin{aligned} Prf_T(x, y) \rightleftharpoons Proof_T(x, y) & \\ & \vee x = 1 \wedge \exists a, b, c < y\ (a + b = c \wedge y = \ulcorner \dot{a} + \dot{b} = \dot{c} \urcorner) \\ & \vee x = 2 \wedge \exists a, b, c < y\ (a \times b = c \wedge y = \ulcorner \dot{a} \times \dot{b} = \dot{c} \urcorner) \\ & \vee x = 3 \wedge \exists a, b < y\ (a = b \wedge y = \ulcorner \dot{a} = \dot{b} \urcorner) \end{aligned} \tag{12}$$

where $Proof_T(x, y)$ denotes the standard Gödel proof predicate for T. Consider evaluations ε_k $(k = 1, \ldots, m)$ extending the standard evaluation ε from (8) as follows:

$$\begin{aligned} &\varepsilon_k W \rightleftharpoons (z = k), \\ &\varepsilon_k r \ \rightleftharpoons 1, \qquad \varepsilon_k p \rightleftharpoons 2, \qquad \varepsilon_k q \rightleftharpoons 3, \\ &\varepsilon_k t \ \rightleftharpoons \mu w.Prf_T(w, \ulcorner k = k \urcorner) \wedge (z = k). \end{aligned}$$

The remaining part of the proof does not change. □

Corollary 4. *The logics $\mathcal{QLP}^c_T(\mathsf{TA})$, $\mathcal{QLP}^c(\mathsf{TA})$ are nonarithmetical. (In fact, the last one is $\Pi^0_1(\mathsf{TA})$-complete). The logics $\mathcal{QLP}^c_T(\mathsf{PA})$, $\mathcal{QLP}^c(\mathsf{PA})$ are Π_2-complete.*

Remark 4. In the proof of theorem 2 the essential point was to construct a proof predicate such that all true formulas of the form $x + y = z$, $xy = z$ and $x = y$ had a common proof. Therefore, this theorem remains true for all classes of proof predicates $\mathcal{K}$ which contain at least one proof predicate of this sort. In case $\mathcal{K}$ contains only proof predicates that imitate real computation processes and does not include predicates of the sort (12) we can prove somewhat weaker results (below) which however suffice to rule out recursive axiomatizability of those logics.

4.2. First order logic of normal proof predicates.

Definition 7. A proof predicate *Prf* is called *normal* if for every $n \in \omega$ the set $\mathrm{Th}(n) = \{x \colon Prf(n, x)\}$ is finite and the function

$$n \mapsto \textit{the Gödel number of } \mathrm{Th}(n)$$

is recursive provably total.

A proof predicate Prf_T for T is *normal multi-conclusion* if any finite set M of theorems of T has a common proof, namely, there exists n such that for every $\varphi \in M$ we have $Prf_T(n, \ulcorner \varphi \urcorner)$.

Remark 5. If a proof predicate $Prf(x, y)$ is normal, then $\mathsf{PA} \vdash$ *"k proves a finite set of theorems"* for every $k \in \omega$, that is, there exists $n \in \omega$ such that

$$\mathsf{PA} \vdash \forall y\ (Prf(k, y) \to y < n).$$

We show that for every class $\mathcal{N}$ of normal multi-conclusion proof predicates the corresponding logic $\mathcal{QLP}^c_{\mathcal{N}}(U)$ is Π_1-hard.

Let $\mathcal{L}_{fin}$ denote the predicate logic of finite models, that is, the set of predicate formulas that are true in all finite models. It is well-known that $\mathcal{L}_{fin}$ is Π_1-complete. We reduce $\mathcal{L}_{fin}$ to $\mathcal{QLP}^c_{\mathcal{N}}(U)$. First we have to prove a kind of arithmetical completeness result for $\mathcal{L}_{fin}$, namely, that $\mathcal{L}_{fin}$ is complete with respect to the class of arithmetical interpretations of the predicate language by formulas that define provably finite or cofinite relations.

Definition 8. Let $\varphi(x_1, \ldots, x_n)$ be an arbitrary arithmetical formula with all free variables shown. We define the following formulas:

$$Fin_\varphi(y) \rightleftharpoons \forall x_1 \ldots x_n\ (\varphi(x_1, \ldots, x_n) \to \bigwedge_{i=1}^{n} x_i \le y),$$
$$St_\varphi(y) \rightleftharpoons Fin_\varphi(y) \lor Fin_{\neg\varphi}(y).$$

Formula φ is called *provably finite* (*provably stable*) if φ is a Δ_1-formula and $\mathsf{PA} \vdash Fin_\varphi(k)$ ($\mathsf{PA} \vdash St_\varphi(k)$ resp.) for some $k \in \omega$.

Let *Fin* and *Stab* denote the classes of all interpretations of the pure predicate language by provably finite and provably stable formulas respectively.

Theorem 3. *Let U be any arithmetical theory,* $\mathsf{PA} \subseteq U \subseteq \mathsf{TA}$. *Then for every predicate formula F*

$$F \in \mathcal{L}_{fin} \iff \forall\alpha \in Fin\ U \vdash \alpha F \iff \forall\alpha \in Stab\ U \vdash \alpha F.$$

Proof. To prove this theorem it suffices to show that

(1) if $F \notin \mathcal{L}_{fin}$, then $\exists\alpha \in Fin\ \mathsf{TA} \nvdash \alpha F$;
(2) if $F \in \mathcal{L}_{fin}$, then $\forall\alpha \in Stab\ \mathsf{PA} \vdash \alpha F$.

The first proposition is obvious. Let us prove the second one.

Step 1. For every formula φ the following formulas are provable in PA:

(1) $St_\varphi(y) \land y < z \to St_\varphi(z)$;
(2) $St_\varphi(y) \land St_\psi(y) \to St_{\varphi\land\psi}(y) \land St_{\varphi\lor\psi}(y) \land St_{\neg\varphi}(y)$;
(3) $St_\varphi(y) \to St_{\exists z\ \varphi}(y)$;
(4) $St_\varphi(y) \to (\exists z\varphi \leftrightarrow \exists z \le (y+1)\ \varphi)$.

Items 1, 2 and 3 are trivial, 4 follows immediately from 3.

Step 2. For every predicate formula F and interpretation $\alpha \in Stab$,

(1) there exists $k \in \omega$ such that $\mathsf{PA} \vdash St_{\alpha F}(k)$;
(2) $\alpha F \in \Delta_1$.

We prove both facts by joint induction on formula F. Induction base when F is an atomic formula holds by the definition of a stable interpretation.

Induction step. Suppose that $F = F_1 \wedge F_2$. Item 2 holds since the class of Δ_1-formulas is closed under boolean connectives. Let us prove 1. By the induction hypothesis, there exist k_1, k_2, such that $\mathsf{PA} \vdash St_{\alpha F_i}(k_i)$ for $i = 1, 2$. Put $k = \max(k_1, k_2)$. From Step 1, (1) and (2), we consequently obtain $\mathsf{PA} \vdash St_{\alpha F_1}(k) \wedge St_{\alpha F_2}(k)$ and $\mathsf{PA} \vdash St_{\alpha(F_1 \wedge F_2)}(k)$. The remaining boolean connectives are treated in a similar way.

Suppose that $F = \exists z G(z, \vec{x})$. From the induction hypothesis it follows that $\alpha G \in \Delta_1$ and there exists $k \in \omega$ such that $\mathsf{PA} \vdash St_{\alpha G}(k)$. Step 1 (3) yields assertion 1. From Step 1 (4) we obtain that $\mathsf{PA} \vdash \exists z\ \alpha G(z, \vec{x}) \leftrightarrow \exists z \leq (k+1)\ \alpha G(z, \vec{x})$. Since the class of Δ_1-formulas is closed under bounded quantifiers, we conclude that $\alpha F \in \Delta_1$.

Step 3. Suppose that $F \in \mathcal{L}_{\mathit{fin}}$ and $\alpha \in \mathit{Stab}$. From the definition it immediately follows that $\mathsf{TA} \vdash \alpha F$. By Step 2, $\alpha F \in \Delta_1$. Therefore $\mathsf{PA} \vdash \alpha F$. □

Theorem 4. *Let $\mathcal{N}$ be any class of normal multi-conclusion proof predicates. Then both logics $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{PA})$ and $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{TA})$ are Π_1-hard and Σ_1-hard.*

Proof. Both logics $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{PA})$ and $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{TA})$ are Σ_1-hard since they are conservative over the predicate calculus PC which is known to be Σ_1-complete. To establish Π_1-hardness we reduce the logic of finite models $\mathcal{L}_{\mathit{fin}}$ (which is Π_1-complete) to both systems. Consider the interpretation *red* of the predicate language in the language $\mathcal{L}^c$ defined in the following way. For every predicate symbol P_i put

$$\mathit{red}\ P_i(\vec{x}) \rightleftharpoons [\![q_i]\!] P_i(\vec{x}).$$

We prove that function *red* performs the reduction of $\mathcal{L}_{\mathit{fin}}$ to $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{PA})$ and $\mathcal{QLP}^c_{\mathcal{N}}(\mathsf{TA})$, that is

$$F \in \mathcal{L}_{\mathit{fin}} \iff \mathit{red}\ F \in \mathcal{QLP}^c_{\mathcal{N}}(\mathsf{PA}) \iff \mathit{red}\ F \in \mathcal{QLP}^c_{\mathcal{N}}(\mathsf{TA}). \tag{13}$$

Step 1. Suppose that $F \in \mathcal{L}_{\mathit{fin}}$ and $* = (\mathit{Prf}, \varepsilon)$ is an arbitrary arithmetical interpretation of the language $\mathcal{L}^c$ with $\mathit{Prf} \in \mathcal{N}$. We define interpretation α of the pure predicate language such that for every predicate symbol P_i

$$\alpha P_i(\vec{x}) \rightleftharpoons (\mathit{red}\ P_i(\vec{x}))^*.$$

Then $\alpha F = (\mathit{red}\ F)^*$ for any predicate formula F. Since the predicate Prf is normal, the interpretation α is provably stable. Thus by theorem 3 we have that $\mathsf{PA} \vdash \alpha F$, whence $\mathsf{PA} \vdash (\mathit{red}\ F)^*$.

Step 2. Suppose that $F \notin \mathcal{L}_{\mathit{fin}}$. Then there exists an interpretation of the pure predicate language $\alpha \in \mathit{Fin}$ such that $\alpha F \notin \mathsf{TA}$. We construct an interpretation $*$ of the language $\mathcal{L}^c$ such that $\mathit{red}\ F^* \notin \mathsf{TA}$.

Let us fix a proof predicate $\mathit{Prf}_T \in \mathcal{N}$. Let $P_1, \ldots P_n$ be the list of all predicate symbols occurring in F. Let M_i be a set consisting of all true formulas of the form $\alpha P_i(\vec{k})$. Since αP_i is a provably finite Δ_1-formula, there exists a number n_i such that $\mathit{Prf}_T(n_i, \varphi)$ holds for every $\varphi \in M_i$. It is also clear that

$$\mathsf{PA} \vdash \alpha P_i(\vec{x}) \leftrightarrow \mathit{Prf}_T(n_i, \ulcorner \alpha P_i(\dot{\vec{x}}) \urcorner). \tag{14}$$

Consider the interpretation $* = (Prf_T, \varepsilon)$, where ε coincides with α on predicate letters and $\varepsilon(q_i) = n_i$. In view of (14), by induction on formula D we can show that $\mathsf{PA} \vdash \alpha D \leftrightarrow (red\, D)^*$. Since $\alpha F \notin \mathsf{TA}$, we conclude $(red\, F)^* \notin \mathsf{TA}$. □

5. Discussion

Though finding a complete axiom system to the first order logic of proofs turned out to be impossible, a more modest goal of finding an exact explicit companion of major first order modal logics, e. g. S4 looks attractive. There are several possible motivations to this problem. In particular, the explicit version of the first order S4 is a step toward finding the BHK semantics for the first order intuitionistic logic, since the Gödel correspondence between intuitioinistic and modal logics can be extended to the first order systems (cf. [10, 18]).

Another natural problem here might be to find an axiomatization of the fragment of the logic of proofs with one individual variable only. The corresponding fragment of the first order provability logic has been shown decidable in [5].

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CUNY Graduate Center, 365 Fifth Avenue, 4319, New York, NY 10016, USA
E-mail address: sartemov@gc.cuny.edu

Department of Mathematics, Moscow State University, Moscow, 119899, Russia
E-mail address: tanya@sidon.math.msu.su