Why Do We Need Justification Logic?

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Abstract

In this paper, we will sketch the basic system of Justification Logic, which is a general logical framework for reasoning about epistemic justification. Justification Logic renders a new, evidence-based foundation for epistemic logic. As a case study, we compare formalizations of the Kripke 'Red Barn' scenario in modal epistemic logic and Justification Logic and show here that the latter provides a deeper analysis. In particular, we argue that modal language fails to fully represent the epistemic closure principle whereas Justification Logic provides its adequate formalization.

1 Introduction

Since Plato, the notion of justification has been an essential component of epistemic studies (cf. [15; 22; 24; 26; 36; 42; 48], and many others). However, until recently, the notion of justification was conspicuously absent in the mathematical models of knowledge within the epistemic logic framework. Commencing from seminal works [28; 52], the notions of Knowledge and Belief have acquired formalization by means of modal logic with atoms *F is known* and *F is believed*. Within this approach, the following analysis was adopted: For a given agent,

$$F \textit{ is known} \quad \sim \quad F \textit{ holds in all epistemically possible situations.}$$

The deficiency of this approach is displayed most prominently, in the *Logical Omniscience* feature of the modal logic of knowledge (cf. [17; 18; 29; 41; 44]). This lack of a justification component has, perhaps, contributed to a certain gap between epistemic logic and mainstream epistemology ([26; 27]). We would like to think that Justification Logic is a step towards filling this void.

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Justification Logic had been anticipated in [23] (as the logic of explicit mathematical proofs) and in [51] (in epistemology), developed in [2; 3; 34; 40] and other papers (as the Logic of Proofs), and then in [4; 5; 6; 8; 12; 20; 21; 25; 31; 33; 43; 45; 47; 53] and other papers in a broader epistemic context. It introduces a long-anticipated mathematical notion of justification, making epistemic logic more expressive. We now have the capacity to reason about justifications, simple and compound. We can compare different pieces of evidence pertaining to the same fact. We can measure the complexity of justifications, which leads to a coherent theory of logical omniscience [7]. Justification Logic provides a novel, evidence-based mechanism of evidence-tracking which seems to be a key ingredient of the analysis of knowledge. Finally, Justification Logic furnishes a new, evidence-based foundation for the logic of knowledge, according to which

$$F \text{ is known} \qquad \sim \qquad F \text{ has an adequate justification.}$$

Justification assertions have the format $t{:}F$, which is read generically as

t is a justification of F.

There is also a more strict 'justificationist' reading in which $t{:}F$ is understood as

t is accepted by agent as a justification of F.

Justification Logic is general enough to incorporate other semantics; e.g., the topological semantics of Justification Logic has been studied in [9].

Justification Logic has been built so far on the simplest base: *classical Boolean logic*, and it is a natural next step to extend these ideas to more elaborate logical models, e.g., intuitionistic and substructural logics, conditionals, relevance logics, and logics of counterfactual reasoning. There are several good reasons for choosing a Boolean logic base for our first meaningful step. At this stage, we are concerned first with *justifications*, which provide a sufficiently serious challenge on even the simplest Boolean base. Once this case is sorted out in a satisfactory way, we can move on to incorporating justifications into other logics. Second, Boolean-based Justification Logic seems to cover known paradigmatic examples, e.g., Russell's and Gettier's examples ([5]) and Kripke's Red Barn Example, which we consider below.

Within the Justification Logic framework, we treat both – **justifications**, which do not necessarily yield the truth of a belief, and **factive justifications**, which yield the truth of the belief. This helps to capture the essence of discussion about these matters in epistemology, where justifications are not generally assumed to be factive.

In this paper, we consider the case of one agent only, although multi-agent Justification Logics have already been studied ([4; 8; 53]).

Formal logical methods do not directly solve philosophical problems but rather provide a tool for analyzing assumptions and ensuring that we draw correct conclusions. Our hope is that Justification Logic does just that.

2 Justifications and Operations

In order to build a formal account of justification, we will make some basic structural assumptions: justifications are abstract objects which have structure, agents do not lose or forget justifications, agents apply the laws of classical logic and accept their conclusions, etc.

We assume two basic operations on justifications, *Application* '$\cdot$' and *Sum* '+,' both having clear epistemic meaning and exact interpretations in relevant mathematical models.

The *Application* operation '$\cdot$' performs one epistemic action, a one-step deduction according to the *Modus Ponens* rule. Application takes a justification s of an implication $F \rightarrow G$ and a justification t of its antecedent, F, and produces a justification $s\cdot t$ of the succedent, G. Symbolically,

$$s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow (s\cdot t){:}G). \tag{1}$$

This is a basic property of justification-type objects assumed in combinatory logic and λ-calculi (cf. [49]), Brouwer-Heyting-Kolmogorov semantics ([50]), Kleene realizability ([30]), the Logic of Proofs LP ([3]), etc. Application principle (1) is related to the epistemic closure principle (cf., for example, [37])

one knows everything that one knows to be implied by what one knows. (2)

However, (1) does not rely on (2), since (1) deals with a broader spectrum of justifications not necessarily linked to knowledge. If justifications s and t are formal Hilbert-style proofs, then $s\cdot t$ can be understood as a new proof obtained from s and t by a single application of the rule *Modus Ponens* to all possible premises $F \rightarrow G$ from s, and F from t:

$$s\cdot t = s * t * \ulcorner G_1 \urcorner * \ldots * \ulcorner G_n \urcorner,$$

where $*$ is concatenation, $\ulcorner X \urcorner$ denotes the Gödel number of X, and G_i's are all formulas from t for which there is a formula $F \rightarrow G_i$ from s.

The second basic operation *Sum* '+' expresses the idea of pooling evidence together without performing any epistemic action. Operation '+' takes justifications s and t and produces $s + t$, which is a justification for everything justified by s or by t.

$$s{:}F \rightarrow (s+t){:}F \quad \text{and} \quad s{:}F \rightarrow (t+s){:}F.$$

In the context of formal proofs, the sum '$s + t$' can be interpreted as a concatenation of proofs s and t

$$s + t = s * t.$$

Such an operation is needed to connect Justification Logic with epistemic modal logic. Justification Logic systems without '+' have been studied in [10; 32; 33].

Justification terms (polynomials) are built from justification variables $x, y, z, \ldots$ and justification constants $a, b, c, \ldots$ by means of the operations '$\cdot$' and '+.' Constants denote atomic justifications which the system no longer analyzes; variables denote unspecified justifications. For the sake of technical convenience, we assume that each constant comes with indices $i = 1, 2, 3 \ldots$ which we will omit whenever it is safe.

More elaborate Justification Logic systems use additional operations on justifications, e.g., verifier '!' and negative verifier '?' ([3; 5; 43; 46; 47]), but we will not need them in this paper.

3 Basic Logic of Justifications

Formulas are built from propositional atoms as the usual formulas of Boolean logic, e.g., by means of logical connectives $\wedge, \vee, \rightarrow, \neg$ with the additional formation rule:

Whenever t is a justification term and F is a formula, $t{:}F$ is again a formula.

The basic Logic of Justifications J_0 contain the following postulates:

A1. *Classical propositional axioms and rule Modus Ponens,*

A2. *Application Axiom* $s{:}(F \rightarrow G) \rightarrow (t{:}F \rightarrow (s \cdot t){:}G)$,

A3. *Sum Axiom* $s{:}F \rightarrow (s+t){:}F,\ s{:}F \rightarrow (t+s){:}F$.

J_0 is the logic of general (not necessarily factive) justifications for an absolutely skeptical agent for whom no formula is provably justified, i.e., J_0 does not derive $t{:}F$ for any t and F. Such an agent is, however, capable of making *relative justification conclusions* of the form

if $x{:}A,\ y{:}B,\ \ldots,\ z{:}C$ hold, then $t{:}F$.

J_0 is able, with this capacity, to adequately emulate other Justification Logic systems within its language.

Well-known examples of epistemic reasoning reveal that logical axioms are often assumed justified. Justification Logic offers a flexible mechanism of *Constant Specifications* that represents different shades of this kind of logical awareness.

Justification Logic distinguishes between assumptions and justified assumptions. Constants are used to denote justifications of assumptions in situations where we don't analyze these justifications further. Suppose we want to postulate that an axiom A is justified for a given agent. The way to state it in Justification Logic is to postulate

$$e_1{:}A$$

for some justification constant e_1 with index 1. Furthermore, if we want to postulate that this new principle $e_1{:}A$ is also justified, we can postulate

$$e_2{:}(e_1{:}A)$$

for the similar constant e_2 with index 2, then

$$e_3{:}(e_2{:}(e_1{:}A)),$$

etc. Using similar constants for 'in-depth justifications' and keeping track of indices is not really necessary, but it is easy and helps in decision procedures (cf. [35]). By $e_n : e_{n-1} : \ldots : e_1 : A$, we mean $e_n : (e_{n-1} : \ldots : (e_1 : A) \ldots)$. A set of assumptions of this kind for a given logic is called a *Constant Specification.* Here is a formal definition.

A **Constant Specification** CS for a given logic $\mathcal{L}$ is a set of formulas

$$e_n\!:\!e_{n-1}\!:\!\ldots\!:\!e_1\!:\!A \quad (n \geq 1),$$

in which A is an axiom of $\mathcal{L}$, and $e_1, e_2, \ldots, e_n$ are similar constants with indices $1, 2, \ldots, n$. We also assume that CS contains all intermediate specifications, i.e., whenever $e_n\!:\!e_{n-1}\!:\!\ldots\!:\!e_1\!:\!A$ is in CS, then $e_{n-1}\!:\!\ldots\!:\!e_1\!:\!A$ is in CS, too. Here are typical examples of constant specifications:

- *empty*: $\mathit{CS} = \emptyset$. This corresponds to an absolutely skeptical agent (cf. a comment after axioms of J_0).
- *finite*: CS is a finite set of formulas. This is a representative case, since any specific derivation in Justification Logic concerns only finite sets of constants and constant specifications.
- *axiomatically appropriate*: For each axiom A, there is a constant e_1 such that e_1:A is in CS, and if $e_n\!:\!\ldots\!:\!e_1\!:\!A \in \mathit{CS}$, then $e_{n+1}\!:\!e_n\!:\!\ldots\!:\!e_1\!:\!A \in \mathit{CS}$.
- *total*: For each axiom A and **any** constants $e_1, e_2, \ldots, e_n$,

  $$e_n\!:\!e_{n-1}\!:\!\ldots\!:\!e_1\!:\!A \in \mathit{CS}.$$

  Naturally, the total constant specification is axiomatically appropriate.

Logic of Justifications with given Constant Specification

$$\mathsf{J}_{\mathit{CS}} = \mathsf{J}_0 + \mathit{CS}.$$

Logic of Justifications

$$\mathsf{J} = \mathsf{J}_0 + \mathrm{R4},$$

where R4 is the **Axiom Internalization Rule**:

For each axiom A and any constants $e_1, e_2, \ldots, e_n$, infer $e_n\!:\!e_{n-1}\!:\!\ldots\!:\!e_1\!:\!A$.

Note that J_0 is $\mathsf{J}_\emptyset$, and J is J_{CS} with the total Constant Specification CS. The latter reflects the idea of the unrestricted logical awareness for J. A similar principle appeared in the Logic of Proofs LP.

For each constant specification CS, J_{CS} enjoys the Deduction Theorem because J_0 contains propositional axioms and *Modus Ponens* as the only rule of inference.

Logical awareness expressed by axiomatically appropriate constant specifications ensures an important *Internalization Property* of the system. This property was anticipated by Gödel in [23] for the logic of explicit mathematical proofs, and was first established for the Logic of Proofs LP in [2; 3].

Theorem 1 *For each axiomatically appropriate constant specification CS,* J_{CS} *enjoys the Internalization Property:*

$$\textit{If} \vdash F, \textit{ then} \vdash p{:}F \textit{ for some justification term } p.$$

Proof. Induction on derivation length. If F is an axiom A, then, since CS is axiomatically appropriate, there is a constant e_1 such that $e_1{:}A$ is in CS, hence an axiom of J_{CS}. If F is in CS, then, since CS is axiomatically appropriate, $e_n{:}F$ is in CS for some constant e_n. If F is obtained by *Modus Ponens* from $X \rightarrow F$ and X, then, by the Induction Hypothesis, $\vdash s{:}(X \rightarrow F)$ and $\vdash t{:}X$ for some s, t. By the Application Axiom, $\vdash (s{\cdot}t){:}F$. □

Internalization in J is an explicit incarnation of the Necessitation Rule in modal logic K:

$$\vdash F \quad \Rightarrow \quad \vdash \Box F.$$

Let us consider some basic examples of derivations in J. In Examples 1 and 2, only constants of level 1 have been used; in such situations we skip indices completely.

Example 1 This example shows how to build a justification of a conjunction from justifications of the conjuncts. In the traditional modal language, this principle is formalized as

$$\Box A \wedge \Box B \rightarrow \Box(A \wedge B).$$

In J we express this idea in a more precise justification language.

1. $A \rightarrow (B \rightarrow A \wedge B)$, a propositional axiom;
2. $c{:}[A \rightarrow (B \rightarrow A \wedge B)]$, from 1, by R4;
3. $x{:}A \rightarrow (c{\cdot}x){:}(B \rightarrow A \wedge B)$, from 2, by A2 and *Modus Ponens*;
4. $x{:}A \rightarrow (y{:}B \rightarrow ((c{\cdot}x){\cdot}y){:}(A \wedge B))$, from 3, by A2 and some propositional reasoning;
5. $x{:}A \wedge y{:}B \rightarrow ((c{\cdot}x){\cdot}y){:}(A \wedge B)$, from 5, by propositional reasoning.

Derived formula 5 contains constant c, which was introduced in line 2, and the complete reading of the result of this derivation is

$$x{:}A \wedge y{:}B \rightarrow ((c{\cdot}x){\cdot}y){:}(A \wedge B), \textit{ given } c{:}[A \rightarrow (B \rightarrow A \wedge B)].$$

Example 2 This example shows how to build a justification of a disjunction from justifications of either disjuncts. In the usual modal language, this is represented by

$$\Box A \vee \Box B \rightarrow \Box(A \vee B).$$

Let us see how this would look in J.

1. $A \rightarrow A \vee B$, by A1;
2. $a{:}[A \rightarrow A \vee B]$, from 1, by R4;
3. $x{:}A \rightarrow (a{\cdot}x){:}(A \vee B)$, from 2, by A2 and *Modus Ponens*;
4. $B \rightarrow A \vee B$, by A1;
5. $b{:}[B \rightarrow A \vee B]$, from 4, by R4;

6. $y{:}B \to (b\cdot y){:}(A \vee B)$ from 5, by A2 and *Modus Ponens*;
7. $(a\cdot x){:}(A \vee B) \to (a\cdot x + b\cdot y){:}(A \vee B)$, by A3;
8. $(b\cdot y){:}(A \vee B) \to (a\cdot x + b\cdot y){:}(A \vee B)$, by A3;
9. $(x{:}A \vee y{:}B) \to (a\cdot x + b\cdot y){:}(A \vee B)$ from 3, 6, 7, 8, by propositional reasoning.

The complete reading of the result of this derivation is

$$(x{:}A \vee y{:}B) \to (a\cdot x + b\cdot y){:}(A \vee B),\ \textit{given}\ a{:}[A \to A \vee B]\ \textit{and}\ \ b{:}[B \to A \vee B].$$

These examples, perhaps, leave the (correct) impression that J can emulate derivations in the corresponding modal logic; here it is K, but at the expense of keeping track of specific justifications. A need for such additional bureaucracy requires explanation and illustration, which is the main goal of this paper. Before we proceed to Section 4, in which such an example is provided, we briefly list applications of Justification Logic so far:

- intended provability semantics for Gödel's provability logic S4 with the Completeness Theorem ([2; 3]);
- formalization of Brouwer-Heyting-Kolmogorov semantics for intuitionistic propositional logic with the Completeness Theorem ([2; 3]);
- a general definition of the Logical Omniscience property and theorems that evidence assertions in Justification Logic are not logically omniscient ([7]);
- an evidence-based approach to Common Knowledge (so-called Justified Common Knowledge) which provides a rigorous epistemic semantics to McCarthy's 'any fool knows' systems ([1; 4; 38]). Justified Common Knowledge offers formal systems which are less restrictive than the usual epistemic logics with Common Knowledge [4];
- formalization of Gettier examples in Justification Logic with missing assumptions and redundancy analysis [5], which demonstrates that Justification Logic methods can be applied in formal epistemology;
- analysis of Knower and Knowability paradoxes ([13; 14]).

The **Correspodence Theorem** ([2; 3; 5; 11; 47]) is a cumulative result stating that for each of major epistemic modal logics K, T, K4, S4, K45, KD45, S5, there is a system of justification terms and a corresponding Justification Logic system (called J, JT, J4, LP, J45, JD45, and JT45) capable of recovering explicit justifications for modalities in any theorem of the original modal logic. This theorem is proven by a variety of methods ranging from cut-elimination in modal logics to semantical proof using Kripke-Fitting models (cf. Section 5).

Complexity issues in Justification Logic have been addressed in [12; 31; 33; 34; 35; 39].

4 Red Barn Example and Tracking Justifications

We illustrate new capabilities of Justification Logic on a paradigmatic Red Barn Example which Kripke developed in 1980 (cf. [37], from which we borrow the formulation, with some editing for brevity).

> Suppose I am driving through a neighborhood in which, unbeknownst to me, papier-mâché barns are scattered, and I see that the object in front of me is a barn. Because I have barn-before-me percepts, I believe that the object in front of me is a barn. Our intuitions suggest that I fail to know barn. But now suppose that the neighborhood has no fake red barns, and I also notice that the object in front of me is red, so I know a red barn is there. This juxtaposition, being a red barn, which I know, entails there being a barn, which I do not, "is an embarrassment"[1].

We consider the Red Barn Example a test for theories that explain knowledge. From such a theory, we expect a way to represent what is happening here which maintains epistemic closure principle (2), but also preserves the epistemic structure of the example.

We present formal analysis of the Red Barn Example in epistemic modal logic (subsections 4.1 and 4.2) and in Justification Logic (subsections 4.3 and 4.4). We will show that epistemic modal logic only indicates that there is a problem, whereas Justification Logic provides resolution.

To make our point, we don't need to formally capture every single detail of the Red Barn story; it suffices to formalize and verify its "entailment" portion. Let

- B be the sentence 'the object in front of me is a barn,'
- R be the sentence 'the object in front of me is red.'

4.1 Red Barn in Modal Logic of Belief

In our first formalization, logical derivation will be performed in epistemic modal logic with 'my belief' modality $\Box$. We then externally interpret some of the occurrences of $\Box$ as 'knowledge' according to the problem's description. In the setting with belief modality $\Box$, episitemic closure principle (2) seems to yield

$$\textit{if } \Box F \textit{ and } \Box(F \rightarrow G) \textit{ are both cases of knowledge, then } \Box G \textit{ is also knowledge.} \qquad (3)$$

The following is a set of natural formal assumptions of the Red Barn Example in the language of epistemic modal logic of belief:

1. $\Box B$, 'I believe that the object in front of me is a barn';
2. $\Box(B \wedge R)$, 'I believe that the object in front of me is a red barn.' At the metalevel, we assume that 2 is knowledge, whereas 1 is not knowledge by the problem's description.

In the basic modal logic of belief K (hence in other modal logics of belief), the following hold:

[1] Dretske [16].

3. $B \wedge R \to B$, as a logical axiom;

4. $\Box(B \wedge R \to B)$, obtaned from 3 by Necessitation. As a logical truth, this also qualifies as knowledge.

Within this formalization, it appears that (3) is violated: line 2, $\Box(B \wedge R)$, and line 4, $\Box(B \wedge R \to B)$ are cases of knowledge whereas $\Box B$ (line 1) is not knowledge. As we see, the modal language here does not help to resolve this issue, but rather obscures its resolution.

4.2 Red Barn in Modal Logic of Knowledge

We will now use epistemic modal logic with 'my knowledge' modality **K**. Here is a straightforward formalization of Red Barn Example assumptions:

1. $\neg \mathbf{K}B$, 'I do not know that the object in front of me is a barn';
2. $\mathbf{K}(B \wedge R)$, 'I know that the object in front of me is a red barn.'

It is easy to see that these assumptions are inconsistent in the modal logic of knowledge. Indeed,

3. $\mathbf{K}(B \wedge R \to B)$, by Necessitation of a propositional axiom;
4. $\mathbf{K}(B \wedge R) \to \mathbf{K}B$, from 3, by modal logic reasoning;
5. $\mathbf{K}B$, from 2 and 4, by *Modus Ponens*.

Lines 1 and 5 formally contradict each other.

Hence, the language of modal logic of knowledge leads to an inconsistent set of formal assumptions and does not reflect the structure of the Red Barn Example properly.

4.3 Red Barn in Justification Logic of Belief

Justification Logic seems to provide a more fine-grained analysis of the Red Barn Example. In Justification Logic, the epistemic closure principle (2) can be naturally formulated according to Application principle (1) as

$$\textit{if } t{:}F \textit{ and } s{:}(F \to G) \textit{ are both cases of knowledge, then } (s \cdot t){:}G \textit{ is also knowledge.} \tag{4}$$

Note that (4) is more precise than (3). In (4), we do not claim that $f(s,t){:}G$ is knowledge for **any** justification $f(s,t)$ but only for a specific $f(s,t)$, which is $s \cdot t$, whereas (3) *de facto* postulates a link between premises $\Box F$, $\Box(F \to G)$ and the conclusion $\Box G$, regardless of how this conclusion was obtained. This is how the ambiguous modal language fails to represent the epistemic closure principle: one cannot claim (3) when justification behind conclusion $\Box G$ is not linked to those behind premises $\Box F$ and $\Box(F \to G)$. This is the essence of the Red Barn example, and a peril which Justification Logic naturally avoids by virtue of its explicit language.

We formalize the Red Barn example in J where $t{:}F$ is interpreted as

'I **believe** F for reason t.'

We naturally introduce individual justifications u for belief that B, and v for belief that $B \wedge R$. The list of assumptions is

1. $u{:}B$, 'u is the reason to believe that the object in front of me is a barn';

2. $v{:}(B \wedge R)$, 'v is the reason to believe that the object in front of me is a red barn.' On the metalevel, the description states that 2 is a case of knowledge, and not merely a belief, whereas 1 is belief which is not knowledge.

Let us try to reconstruct the reasoning of the agent in J:

3. $B \wedge R \rightarrow B$, logical axiom;
4. $a{:}[B \wedge R \rightarrow B]$, from 3, by Axiom Internalization. This is also a case of knowledge;
5. $v{:}(B \wedge R) \rightarrow (a \cdot v){:}B$, from 4, by Application and *Modus Ponens*;
6. $(a \cdot v){:}B$, from 2 and 5, by *Modus Ponens*.

Closure holds! By reasoning in J, we have concluded that $(a \cdot v){:}B$ is a case of knowledge, i.e., 'I know B for reason $a \cdot v$.' The fact that $u{:}B$ is not a case of knowledge does not spoil the closure principle, since the latter claims knowledge specifically for $(a \cdot v){:}B$. Hence, after observing a red façade, I indeed know B, but this knowledge has nothing to do with 1, which remains a case of belief rather than of knowledge, and Justification Logic formalization represents this fairly.

4.4 Red Barn in Justification Logic of Knowledge

Within this formalization, $t{:}F$ is interpreted as

'I **know** F for reason t.'

As in Section 4.2, we assume

1. $\neg u{:}B$, 'u is not a sufficient reason to know that the object is a barn';
2. $v{:}(B \wedge R)$, 'v is a sufficient reason to know that the object is a red barn.'

This is a perfectly consistent set of assumptions even in the logic of factive justifications

$$\mathsf{J} + \textit{Factivity Principle } (t{:}F \rightarrow F).$$

As in 4.3, we can derive $(a \cdot v){:}B$ where $a{:}[B \wedge R \rightarrow B]$, but this does not lead to a contradiction. Claims $\neg u{:}B$ and $(a \cdot v){:}B$ naturally co-exist. They refer to different justifications u and $a \cdot v$ of the same fact B; one of them insufficient and the other quite sufficient for my knowledge that B.

4.5 Red Barn and Formal Epistemic Models

It appears that in 4.3 and 4.4, Justification Logic represents the structure of the Red Barn Example in a reasonable way which was not directly captured by epistemic modal logic.

I all fairness to modal tools, we could imagine a formalization of the Red Barn Example in a sort of bi-modal language with distinct modalities for knowledge and belief, where both claims hold: '$\Box B$,' by perceptual belief that B, and '$\mathbf{K}B$' for knowledge that B which is logically derived from perceptual knowledge $\mathbf{K}(B \wedge R)$. However, it seems that such a resolution will, intellectually, involve repeating Justification Logic arguments from 4.3 and 4.4 in a way that obscures, rather than reveals, the truth. Such a bi-modal formalization would distinguish $u{:}B$ from $(a \cdot v){:}B$ not because they have different reasons (which reflects the true epistemic structure of the problem), but rather because the former is labelled 'belief' and the latter 'knowledge.' But what if we need to keep track of different unrelated reasons which are all cases of either knowledge or belief? Following this multi-modal approach, we will likely end up with a collection of distinct modalities, each for different reasons, as well as a mounting pile of additional assumptions concerning these modalities – all just to avoid revealing the justification structure of a problem which can easily fit into a very basic justification logic J with the bare minimum of epistemological assumptions.

5 Basic Epistemic Semantics

This section will provide the basics of epistemic semantics for Justification Logic, the main ideas of which have been suggested by Fitting in [20]. The standard epistemic semantics for J (cf. [5]) has been provided by the proper adaptation of Kripke-Fitting models [20] and Mkrtychev models [40].

A Kripke-Fitting **J-model** $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$ is a Kripke model $(W, R, \Vdash)$ enriched with an **admissible evidence function** $\mathcal{E}$ such that $\mathcal{E}(t, F) \subseteq W$ for any justification t and formula F. Informally, $\mathcal{E}(t, F)$ specifies the set of possible worlds where t is considered admissible evidence for F. The intended use of $\mathcal{E}$ is in the truth definition for justification assertions:

$u \Vdash t{:}F$ if and only if

1. *F holds for all possible situations, i.e., $v \Vdash F$ for all v such that uRv;*

2. *t is an admissible evidence for F at u, i.e., $u \in \mathcal{E}(t, F)$.*

An admissible evidence function $\mathcal{E}$ must satisfy the closure conditions with respect to operations '$\cdot$' and '$+$':

- *Application*: $\mathcal{E}(s, F \to G) \cap \mathcal{E}(t, F) \subseteq \mathcal{E}(s{\cdot}t, G)$. This condition states that whenever s is an admissible evidence for $F \to G$ and t is an admissible evidence for F, their 'product,' $s{\cdot}t$, is an admissible evidence for G.

- *Sum*: $\mathcal{E}(s, F) \cup \mathcal{E}(t, F) \subseteq \mathcal{E}(s{+}t, F)$. This condition guarantees that $s{+}t$ is an admissible evidence for F whenever either s is an admissible evidence for F or t is an admissible evidence for F.

Given a model $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$, the forcing relation $\Vdash$ is extended from sentence variables to all formulas as follows: for each $u \in W$,

1. $\Vdash$ respects Boolean connectives at each world ($u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$; $u \Vdash \neg F$ iff $u \not\Vdash F$, etc.);

2. $u \Vdash t{:}F$ iff $u \in \mathcal{E}(t, F)$ and $v \Vdash F$ for every $v \in W$ with uRv.

Note that an admissible evidence function $\mathcal{E}$ may be regarded as a Fagin-Halpern awareness function [19] equipped with the structure of justifications.

A model $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$ *respects a Constant Specification CS at* $u \in W$ if $u \in \mathcal{E}(c, A)$ for all formulas $c{:}A$ from CS. Furthermore, $\mathcal{M} = (W, R, \mathcal{E}, \Vdash)$ *respects a Constant Specification CS* if $\mathcal{M}$ respects CS at each $u \in W$.

Theorem 2 *For any Constant Specification CS,* J_{CS} *is sound and complete for the class of all Kripke-Fitting models respecting CS.*

Mkrtychev semantics is a predecessor of Kripke-Fitting semantics ([40]). *Mkrtychev models* are Kripke-Fitting models with a single world, and the proof of Theorem 2 can be easily modified to establish completeness of J_{CS} with respect to Mkrtychev models.

Theorem 3 *For any Constant Specification CS,* J_{CS} *is sound and complete for the class of Mkrtychev models respecting CS.*

Theorem 3 shows that the information about Kripke structure in Kripke-Fitting models can be completely encoded by the admissible evidence function. Mkrtychev models play an important theoretical role in Justification Logic [12; 31; 34; 39]. On the other hand, Kripke-Fitting models can be useful as counter-models with desirable properties since they take into account both epistemic Kripke structure and evidence structure. Speaking metaphorically, Kripke-Fitting models naturally reflect two reasons why a certain fact F can be unknown to an agent: F fails at some possible world or an agent does not have a sufficient evidence of F.

Another application area of Kripke-Fitting style models is Justification Logic with both epistemic modalities and justification assertions (cf. [4; 8]).

6 Adding Factivity

Factivity states that a given justification of F is factive, i.e., sufficient for an agent to conclude that F is true. The corresponding *Factivity Axiom* claims that justifications are factive:

$$t{:}F \rightarrow F,$$

which has a similar motivation to the Truth Axiom in epistemic modal logic

$$\mathbf{K}F \rightarrow F,$$

widely accepted as a basic property of knowledge.

The Factivity Axiom first appeared in the Logic of Proofs LP as a principal feature of mathematical proofs. Indeed, in this setting Factivity is valid: if there is a mathematical proof t of F, then F must be true.

We adopt the Factivity Axiom for justifications that lead to knowledge. However, factivity alone does not warrant knowledge, which has been demonstrated by Gettier examples ([22]).

Logic of Factive Justifications:

$$\mathsf{JT}_0 = \mathsf{J}_0 + \textit{Factivity Axiom},$$

$$\mathsf{JT} = \mathsf{J} + \textit{Factivity Axiom}.$$

Systems JT_{CS} corresponding to Constant Specifications CS are defined similarly to J_{CS}.

JT-models are J-models with reflexive accessibility relations R. The reflexivity condition makes each possible world accessible from itself, which exactly corresponds to the Factivity Axiom. The direct analogue of Theorem 1 hold for JT_{CS} as well.

Theorem 4 *For any Constant Specification CS, each of the logics* JT_{CS} *is sound and complete with respect to the class of* JT*-models respecting CS.*

Mkrtychev JT**-models** are singleton JT-models, i.e., JT-models with singleton W's.

Theorem 5 *For any Constant Specification CS, each of the logics* JT_{CS} *is sound and complete with respect to the class of Mkrtychev* JT*-models respecting CS.*

7 Conclusions

Modal logic fails to fully represent the epistemic closure principle whereas Justification Logic provides a more adequate formalization.

Justification Logic extends the logic of knowledge by the formal theory of justification. Justification Logic has roots in mainstream epistemology, mathematical logic, computer science, and artificial intelligence. It is capable of formalizing a significant portion of reasoning about justifications.

It remains to be seen to what extent Justification Logic can be useful for analysis of empirical, perceptual, and *a priori* types of knowledge. From the perspective of Justification Logic, such knowledge may be considered as justified by constants (i.e., atomic justifications). Apparently, further discussion is needed here.

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