Modal Logics and Topological Semantics for Hybrid Systems

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Abstract

In this paper, we introduce the logic of a control action **S4F** and the logic of a continuous control action **S4C** on the state space of a dynamical system. The state space here is represented by a topological space $(X, \mathcal{T})$ and the control action by a function f from X to X. We present an intended topological semantics and a Kripke semantics, give both a Hilbert-style and Gentzen-style axiomatization for **S4F** and **S4C**, prove completeness with respect to both semantics as well as a cut-elimination for the corresponding sequent calculi and show the logics to be decidable.

1 Introduction

Let $\mathcal{L}_\Box$ be the propositional modal language generated from a countable set **PV** of propositional variables, the propositional constant $\bot$ (falsum), the propositional connective $\rightarrow$ (implication), and the modal operator $\Box$. Let $\mathcal{L}_{\Box a}$ be the propositional language extending $\mathcal{L}_\Box$ which includes, in addition, a new modal operator $[a]$.

Let **S4** denote the subset of $\mathcal{L}_{\Box a}$ consisting of all formulas derivable from a standard axiomatization of classical propositional logic together with the axiom schemes: $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$, $\Box\varphi \rightarrow \varphi$ and $\Box\varphi \rightarrow \Box\Box\varphi$, using the inference rules of modus ponens (MP) and $\Box$-necessitation.

We develop a bimodal extension of **S4**, which we call **S4F**, in the language $\mathcal{L}_{\Box a}$ with the single new modal operator $[a]$. In the intended topological semantics for this new logic, the **S4** modality $\Box$ is interpreted in the standard way as the topological interior operator, and $[a]$ is interpreted as the inverse image $f^{-1}(\cdot)$ for a fixed total function $f : X \rightarrow X$ on the state space X, equipped with a topology $\mathcal{T}$. For each propositional formula φ of $\mathcal{L}_{\Box a}$, $\|\varphi\|$ is a subset of X, and $\|[a]\varphi\|$ is the set of points $x \in X$ such that after applying the function $f : X \rightarrow X$ interpreting a, we have $f(x) \in \|\varphi\|$. So $\|[a]\varphi\| = f^{-1}(\|\varphi\|)$. The set map f^{-1} commutes with all the Boolean operations on sets and the axiom schemes for **S4F** reflect this: $[a](\varphi \rightarrow \psi) \rightarrow ([a]\varphi \rightarrow [a]\psi)$ and $\neg[a]\varphi \leftrightarrow [a]\neg\varphi$. The $[a]$-necessitation inference rule corresponds to the totality of f.

When the function f is continuous with respect to the topology $\mathcal{T}$, $\|[a]\varphi\|$ is an open set (closed set) whenever $\|\varphi\|$ is open (closed), and f *is* continuous exactly when the formula

$$[a]\Box\varphi \rightarrow \Box[a]\varphi$$

is satisfied (evalutes as the whole space X) for each $\varphi \in \mathcal{L}_{\Box a}$.

In application to continuous dynamics in hybrid control systems, we think of the symbol "a" as denoting a "control action", typically a vector field applied for a fixed duration, so that the function f interpreting a is a section of a *flow* on the state space (manifold).

In dynamic or program logics (see, for eg. [Ha84] or [KT90]), formulas of the form

$$\varphi \rightarrow [p]\psi$$

where p denotes a program, express the Hoare partial correctness assertion $\{\varphi\}\, p\, \{\psi\}$: "if program p begins execution in a φ state then it will terminate in a ψ state". In the logic **S4F**, formulas of the form:

$$\psi \to [a]\varphi$$

can be read as: "whenever ψ, then action a always makes it the case that φ" or more succinctly, "action a always takes ψ states to φ states". Such a formula is true (evaluates as the whole space) in a topological model $\mathfrak{T} = (X, \mathcal{T}, f; \xi)$ exactly when, for all $x \in X$:

$$x \in \|\psi\|_\xi \quad \text{implies} \quad f(x) \in \|\varphi\|_\xi$$

where ξ is a valuation of atomic propositions as subsets of X. More generally,

$$\psi \to [a]^k\varphi$$

reads "k iterations of action a always takes ψ states to φ states", where $[a]^0\varphi$ is just φ and $[a]^{k+1}\varphi$ is $[a][a]^k\varphi$.

In this paper, we concentrate on the (classical) logic of a single control action. We present a topological semantics and a Kripke semantics, give both a Hilbert-style axiomatization and a Gentzen sequent calculus for the logic **S4F**, prove completeness with respect to both semantics as well as a semantic proof of cut-elimination for the sequent calculus and show the logic to be decidable.

2 Syntax and Topological Semantics

Definition 2.1 *Let $\mathcal{L}_{\Box a}$ be the propositional language generated from a countable set AP of atomic propositions, the propositional constant $\bot$ (falsum), the propositional connective $\to$ (implication), and the modal operators $\Box$ and $[a]$.*

Within the language $\mathcal{L}_{\Box a}$, we can define in the usual way the propositional constants and the other classical propositional connectives in terms of $\bot$ and $\to$, the diamond operators $\Diamond$ and $\langle a\rangle$ as the classical duals of $\Box$ and $[a]$, respectively:

$$\begin{aligned}
\top &\doteq \neg\bot \\
\neg\varphi &\doteq \varphi \to \bot \\
\varphi \wedge \psi &\doteq \neg(\varphi \to \neg\psi) \\
\varphi \vee \psi &\doteq \neg\varphi \to \psi \\
\varphi \leftrightarrow \psi &\doteq (\varphi \to \psi) \wedge (\psi \to \varphi) \\
\Diamond\varphi &\doteq \neg\Box\neg\varphi \\
\langle a\rangle\varphi &\doteq \neg[a]\neg\varphi
\end{aligned}$$

Definition 2.2 *A* topological structure *for the propositional language $\mathcal{L}_{\Box a}$ is a triple $\mathfrak{T} = (X, \mathcal{T}, f)$ where*

- *$X \neq \emptyset$ is the state space;*
- *$\mathcal{T} \subseteq \mathcal{P}(X)$ is a topology on X (i.e. $\emptyset, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary unions and finite intersections); and*
- *$f : X \to X$ is a total function.*

Note that at this stage, f is not assumed to be anything other than total; in particular, it is not assumed to be continuous w.r.t. $\mathcal{T}$.

Definition 2.3 *A* valuation *for a topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ is any map $\xi : AP \to \mathcal{P}(X)$ assigning a subset $\xi(p) \subseteq X$ to each $p \in AP$. Each such valuation uniquely extends to a valuation map $\|\cdot\|_\xi : \mathcal{L}_{\Box a} \to \mathcal{P}(X)$, satisfying the following clauses:*

$$
\begin{array}{rcl}
\|p\|_\xi & = & \xi(p) \\
\|\bot\|_\xi & = & \emptyset \\
\|\varphi \to \psi\|_\xi & = & -\|\varphi\|_\xi \cup \|\psi\|_\xi \\
\|\Box\varphi\|_\xi & = & int_{\mathcal{T}}\left(\|\varphi\|_\xi\right) \\
\|[a]\varphi\|_\xi & = & f^{-1}\left(\|\varphi\|_\xi\right)
\end{array}
$$

where $int_{\mathcal{T}}$ is the interior operator *determined by the topology $\mathcal{T}$, i.e. for all $A \subseteq X$,*

$$int_{\mathcal{T}}(A) \quad = \quad \bigcup\{U \in \mathcal{T} \mid U \subseteq A\}$$

and f^{-1} is the inverse-image operator *determined by the total function f:*

$$f^{-1}(A) \quad = \quad \{x \in X \mid f(x) \in A\}$$

Definition 2.4 *A* topological model *for $\mathcal{L}_{\Box a}$ is a pair $(\mathfrak{T}, \xi)$, where $\mathfrak{T} = (X, \mathcal{T}, f)$ is a topological structure for $\mathcal{L}_{\Box a}$ and $\xi : AP \to \mathcal{P}(X)$ is a valuation for $\mathfrak{T}$.*

Definition 2.5 *Let $\varphi \in \mathcal{L}_{\Box a}$ be a propositional formula.*

- *φ is* satisfied *at a state $x \in X$ in a topological model $(\mathfrak{T}, \xi)$ iff $x \in \|\varphi\|_\xi$.*
- *φ is* true *in a topological model $(\mathfrak{T}, \xi)$, written $(\mathfrak{T}, \xi) \models \varphi$, iff $\|\varphi\|_\xi = X$;*
- *φ is* valid *in a topological structure $\mathfrak{T}$, written $\mathfrak{T} \models \varphi$, iff for* all *valuations ξ for $\mathfrak{T}$, we have $\|\varphi\|_\xi = X$;*

- φ *is* topologically valid *iff* $\mathfrak{T} \models \varphi$ *for every topological structure* $\mathfrak{T} = (X, \mathcal{T}, f)$ *for* $\mathcal{L}_{\Box a}$.

The topological semantics for the defined constants, connectives and modal operators are as one would expect.

$$\begin{array}{rcl} \|\top\|_\xi & = & X \\ \|\neg\varphi\|_\xi & = & -\|\varphi\|_\xi \\ \|\varphi \wedge \psi\|_\xi & = & \|\varphi\|_\xi \cap \|\psi\|_\xi \\ \|\varphi \vee \psi\|_\xi & = & \|\varphi\|_\xi \cup \|\psi\|_\xi \\ \|\Diamond\varphi\|_\xi & = & cl_{\mathcal{T}}\left(\|\varphi\|_\xi\right) \\ \|\langle a\rangle\varphi\|_\xi & = & -f^{-1}\left(-\|\varphi\|_\xi\right) \end{array}$$

where $cl_{\mathcal{T}}$ is the closure operator determined by the topology $\mathcal{T}$, i.e. for any $A \subseteq X$,

$$cl_{\mathcal{T}}(A) \quad = \quad \bigcap\{C \mid -C \in \mathcal{T} \text{ and} A \subseteq C\}$$

Observe that for any topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ and valuation ξ for $\mathfrak{T}$,

$$\|\varphi \to \psi\|_\xi = X \quad \text{iff} \quad \|\varphi\|_\xi \subseteq \|\psi\|_\xi$$

More generally,

$$\|\varphi \to \psi\|_\xi = \{x \in X \mid \text{ if } x \in \|\varphi\|_\xi \text{ then } x \in \|\psi\|_\xi\}$$

The proposed reading of formulas of the form:

$$\psi \to [a]\varphi$$

as "action a always takes ψ states to φ states" is based on the fact that in any topological model $(\mathfrak{T}, \xi)$,

$$(\mathfrak{T}, \xi) \models \psi \to [a]\varphi \quad \text{iff} \quad \text{forall} x \in X, \text{if} x \in \|\psi\|_\xi \text{ then} f(x) \in \|\varphi\|_\xi .$$

We can embed Intuitionistic propositional logic **Int** within **S4** via the standard Gödel translation by "Boxing" all propositional variables, i.e. $\Box p$, and defining Intuitionistic negation $\sim$ and Intuitionistic implication $\rightsquigarrow$ as:

$$\sim \varphi \triangleq \Box(\neg\varphi)$$
$$\varphi \rightsquigarrow \psi \triangleq \Box(\varphi \to \psi)$$

Topologically, this means that in the Intuitionistic semantics, all propositional variables denote open sets, Intuitionistic negation corresponds to the interior of the complement, and Intuitionistic implication corresponds to the interior of classical implication.

3 Hilbert-style Axiomatization

Definition 3.1 *The Hilbert-style proof system for the logic* **S4F** *has the following axiom schemes, in the language* $\mathcal{L}_{\Box a}$*:*

$$
\begin{array}{ll}
\mathbf{CP}: & \text{axioms of classical propositional logic in } \mathcal{L}_{\Box a} \\
\Box\mathbf{K}: & \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \\
\Box\mathbf{T}: & \Box\varphi \to \varphi \\
\Box\mathbf{4}: & \Box\varphi \to \Box\Box\varphi \\
[a]\mathbf{K}: & [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi) \\
[a]\neg: & [a]\neg\varphi \leftrightarrow \neg[a]\varphi
\end{array}
$$

and the inference rules:

$$
\begin{array}{ll}
\textit{modus ponens}: & \dfrac{\varphi,\ \varphi \to \psi}{\psi} \\
\Box\ -\textit{necessitation}\ : & \dfrac{\varphi}{\Box\varphi} \\
[a]\ -\textit{necessitation}\ : & \dfrac{\varphi}{[a]\varphi}
\end{array}
$$

We write

$$\mathbf{S4F} \vdash_H \varphi$$

or say φ *is* $\mathbf{S4F}_H$ *provable, if the formula* $\varphi \in \mathcal{L}_{\Box a}$ *has an* **S4F** *Hilbert-style derivation.*

The axiom schemes $\Box\mathbf{K}$, $\Box\mathbf{T}$ and $\Box\mathbf{4}$, together with **CP**, and the rules of modus ponens and $\Box$-necessitation, constitute the standard Hilbert-style proof system for propositional **S4**. From McKinsey and Tarski [McK41], [MT44], the **S4** axioms are true in every topological space $(X, \mathcal{T})$ and hence true in every topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$, and the inference rules are truth-preserving (i.e. if the hypotheses evaluate as the whole space X, then so does the conclusion).

The axioms $[a]\mathbf{K}$ and $[a]\neg$ for the $[a]$ modality, together with the $[a]$-necessitation rule, can be found in [Lem77][1], where the uni-modal logic is given the name **KF** ("**F**" for "function"). The logic **KF** is identified as characteristic for total (serial) and functional (deterministic) binary relations in the Kripke semantics. In a sense, the $[a]$ operator is nothing more than the "next-time" or "next-state" modality of temporal logics[2], given a more abstract semantics.

[1]The source manuscript of the "Lemmon Notes" [Lem77] is dated 1966, and was a collaboration of E. J. Lemmon and Dana Scott. It was edited for [Lem77] by Krister Segerberg.

[2]The first appearance of the **KF** axioms seems to be in A. N. Prior's [Pri57] as the axioms for the "tomorrow it will be the case that" modality, and appear again in that guise in [Seg67]. See also Appendix B of Prior's [Pri67].

The novelty here lies in combining it with the **S4** $\Box$ and $\Diamond$ modalities to give symbolic representation to a topology as well as an arbitrary function.

The converse of $[a]\mathbf{K}$ is derivable as follows:

1.	$[a]\varphi \to [a]\psi$	hypothesis
2.	$\neg[a]\varphi \vee [a]\psi$	from 1. by propositional logic
3.	$[a]\neg\varphi \vee [a]\psi$	from 2. by $[a]\neg$ and propositional logic
4.	$\neg\varphi \to (\varphi \to \psi)$	tautology of propositional logic
5.	$[a]\,(\neg\varphi \to (\varphi \to \psi))$	from 4. by $[a]-$necessitation
6.	$[a]\neg\varphi \to [a](\varphi \to \psi)$	from 5. by $[a]\mathbf{K}$
7.	$\psi \to (\varphi \to \psi)$	tautology of propositional logic
8.	$[a]\,(\psi \to (\varphi \to \psi))$	from 7. by $[a]-$necessitation
9.	$[a]\psi \to [a](\varphi \to \psi)$	from 8. by $[a]\mathbf{K}$
10.	$[a](\varphi \to \psi)$	from 3.,6. and 9. by propositional logic

Hence $[a]$ commutes with each of the classical (Boolean) propositional connectives. Thus as a modal operator, $[a]$ is classically self-dual, since in $\mathbf{S4F}_H$,

$$\langle a\rangle\varphi \;\leftrightarrow\; \neg[a]\neg\varphi \;\leftrightarrow\; \neg\neg[a]\varphi \;\leftrightarrow\; [a]\varphi$$

The following are $\mathbf{S4F}_H$ provable, for any formulas $\varphi, \psi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$, where if $k > 0$, $[a]^k\varphi$ denotes the formula $[a][a]...[a]\varphi$, with k iterations of the $[a]$ operator and if $k = 0$, then $[a]^k\varphi$ is just φ.

$$\begin{array}{ll}
[a]^k\neg: & \neg[a]^k\varphi \leftrightarrow [a]^k\neg\varphi \\
[a]^k\to: & [a]^k(\varphi \to \psi) \leftrightarrow ([a]^k\varphi \to [a]^k\psi) \\
[a]^k\wedge: & [a]^k(\varphi \wedge \psi) \leftrightarrow ([a]^k\varphi \wedge [a]^k\psi) \\
[a]^k\vee: & [a]^k(\varphi \vee \psi) \leftrightarrow ([a]^k\varphi \vee [a]^k\psi) \\
[a]^k\top: & [a]^k\top \\
[a]^k\bot: & [a]^k\bot \leftrightarrow \bot \\
[a]^k\Box: & [a]^k\Box\varphi \to [a]^k\varphi \\
[a]^k\Diamond: & [a]^k\varphi \to [a]^k\Diamond\varphi
\end{array}$$

The following are admissible inference rules in $\mathbf{S4F}_H$, for any formulas $\varphi, \psi, \chi \in \mathcal{L}_{\Box a}$ and $k, l \in \mathbb{N}$:

$$[a]^k\textit{-necessitation}: \quad \frac{\varphi}{[a]^k\varphi}$$

$$\textit{Monotonicity of } [a]^k: \quad \frac{\varphi \to \psi}{[a]^k\varphi \to [a]^k\psi}$$

$$\textit{Hoare composition}: \quad \frac{\varphi \to [a]^k\chi,\ \chi \to [a]^l\psi}{\varphi \to [a]^{k+l}\psi}$$

Observe that there are no axioms for **S4F** containing both $\Box$ and $[a]$, so the behaviors of the two modalities are quite independent and the logic can be thought of as a "direct product" of **S4** and **KF**. When we adjoin a true bimodal axiom such as

$$\textbf{Cont}: \quad [a]\Box\varphi \to \Box[a]\varphi$$

the result is a richer "amalgamated product" of **S4** and **KF**.

Proposition 3.2 Topological Soundness of **S4F** Hilbert-style axiomatization
For all formulas φ of $\mathcal{L}_{\Box a}$, if $\textbf{S4F} \vdash_H \varphi$ *then φ is topologically valid.*

Proof. The topological validity of the **S4** axioms for $\Box$ plus the validity-preservation of modus ponens $\Box$-necessitation follow trivially from the properties of the interior operator; see [McK41], [MT44]. The semantical validity of the $[a]$-necessitation rule translates as

$$\|\varphi\|_\xi = X \text{ implies } f^{-1}\left(\|\varphi\|_\xi\right) = X$$

and the equation $f^{-1}(X) = X$ holds exactly when $f : X \to X$ is a *total* function. The validity of the **F** axioms for $[a]$ are immediate from the properties of the inverse-image operator. ■

4 Sequent Calculus

We give a Gentzen-style sequent calculus for the logic **S4F**. In the following, φ and ψ are arbitrary formulas of the language $\mathcal{L}_{\Box a}$ and Γ and Δ (with or without subscripts) are (possibly empty) *multisets* of formulas of $\mathcal{L}_{\Box a}$ (i.e. finite "sets" in which repetitions are allowed, so we can ignore the Exchange rules required in Gentzen systems that treat sequences of formulas rather than multisets). The join or union of two multisets Γ and Δ is written Γ, Δ, and either Γ, φ or φ, Γ denote the multiset resulting from the join of Γ and the multiset whose sole member is φ. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$; the multiset Γ on the left is called the *antecedent*, and the multiset Δ on the right is called the *succedent*.

If multisets of formulas Γ and Δ are $\{\{\varphi_1, ..., \varphi_n\}\}$ and $\{\{\psi_1, ..., \psi_m\}\}$, respectively, then the sequent $\Gamma \Rightarrow \Delta$ translates as the propositional formula

$$(\varphi_1 \wedge ... \wedge \varphi_n) \to (\psi_1 \vee ... \vee \psi_m)$$

of $\mathcal{L}_{\Box a}$, and is abbreviated as:

$$\bigwedge \Gamma \to \bigvee \Delta$$

In addition, we use $\Box\Gamma$ and $[a]\Gamma$ as abbreviations for the multisets

$$\{\{\ \Box\varphi_1, ..., \Box\varphi_n\ \}\} \text{ and } \{\{\ [a]\varphi_1, ..., [a]\varphi_n\ \}\}$$

respectively.

Definition 4.1 *The Gentzen-style sequent calculus for the logic* **S4F** *has the following axioms and rules.*

1. ***Classical propositional logic axioms and rules for*** $\{\bot, \to\}$***:***

$$(Axiom): \ \varphi \Rightarrow \varphi \qquad (\bot \Rightarrow): \ \bot \Rightarrow$$

$$(\to\Rightarrow): \ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \psi, \Gamma_2 \Rightarrow \Delta_2}{\varphi \to \psi, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\Rightarrow\to): \ \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

2. ***Structural rules:***

$$(\ Weak \Rightarrow): \ \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \ Weak): \ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(Contr \Rightarrow): \ \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \ Contr): \ \frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}$$

$$(Cut): \ \frac{\Gamma_1 \Rightarrow \Delta_1, \varphi \quad \varphi, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

3. **S4** ***rules for*** $\Box$***:***

$$(\Box \Rightarrow): \ \frac{\varphi, \Gamma \Rightarrow \Delta}{\Box\varphi, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \Box): \ \frac{\Box\Gamma \Rightarrow \varphi}{\Box\Gamma \Rightarrow \Box\varphi}$$

4. **KF** ***rule for*** $[a]$***:***

$$([a] \Rightarrow [a]): \quad \frac{\Gamma \Rightarrow \Delta}{[a]\Gamma \Rightarrow [a]\Delta}$$

We write

$$\mathbf{S4F} \vdash_G \Gamma \Rightarrow \Delta$$

if the sequent $\Gamma \Rightarrow \Delta$ *in the language* $\mathcal{L}_{\Box a}$ *has a* **S4F** *sequent calculus derivation, and we write*

$$\mathbf{S4F} \vdash_G^- \Gamma \Rightarrow \Delta$$

if the sequent $\Gamma \Rightarrow \Delta$ *in the language* $\mathcal{L}_{\Box a}$ *has a* cut-free **S4F** *sequent calculus derivation.*

Note, that ***1., 2., 3.*** give a Gentzen-style axiomatization of **S4** (cf. [TS96]).

Proposition 4.2 Equivalence of Sequent Calculus and Hilbert-style proof system for **S4F**
Let Γ *and* Δ *be multisets of formulas of* $\mathcal{L}_{\Box a}$*, and let* φ *be any formula of* $\mathcal{L}_{\Box a}$*.*

(i) *If* **S4F** $\vdash_G \Gamma \Rightarrow \Delta$ *then* **S4F** $\vdash_H \bigwedge \Gamma \to \bigvee \Delta$.

(ii) *If* **S4F** $\vdash_H \varphi$ *then* **S4F** $\vdash_G \Rightarrow \varphi$.

Proof. (i) Proceed by induction on the complexity of the $\mathbf{S4F}_G$ sequent calculus derivation of $\Gamma \Rightarrow \Delta$. Since ***1., 2., 3.*** axiomatize a sequent variant of **S4** it suffices to verify the rule concerning the modality $[a]$.

So assume the *last* rule applied in the derivation of $\Gamma \Rightarrow \Delta$ is $([a] \Rightarrow [a])$, and the result holds for the upper sequent of the rule: Γ is $[a]\Gamma'$ and Δ is $[a]\Delta'$, and the sequent $\Gamma \Rightarrow \Delta$ is derived from $\Gamma' \Rightarrow \Delta'$ by the $([a] \Rightarrow [a])$ rule. By the induction hypothesis, **S4F** $\vdash_H \bigwedge \Gamma \to \bigvee \Delta$. Then

1.	$\bigwedge \Gamma' \to \bigvee \Delta'$	induction hypothesis
2.	$[a]\,(\bigwedge \Gamma' \to \bigvee \Delta')$	from 1. by $[a]-$necessitation
3.	$[a] \bigwedge \Gamma' \to [a] \bigvee \Delta'$	from 2. by $[a]\mathbf{K}$
4.	$[a] \bigwedge \Gamma' \leftrightarrow \bigwedge [a]\Gamma'$	theorem of $\mathbf{S4F}_H$
5.	$[a] \bigvee \Delta' \leftrightarrow \bigvee [a]\Delta'$	theorem of $\mathbf{S4F}_H$
6.	$\bigwedge [a]\Gamma' \to \bigvee [a]\Delta'$	from 3.,4. and 5. by propositional logic

(ii) We show that each of the axioms of $\mathbf{S4F}_H$ are derivable in $\mathbf{S4F}_G$, and that each of the inference rules of $\mathbf{S4F}_H$ are preserved in $\mathbf{S4F}_G$. For the axioms and rules of **S4** this is known (cf. [TS96]).

Consider $[a]$-*necessitation*. Assume **S4F** $\vdash_G \Rightarrow \varphi$. Then applying $([a] \Rightarrow [a])$ (with empty antecedent) we obtain **S4F** $\vdash_G \Rightarrow [a]\varphi$.

Axiom $[a]\mathbf{K}$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \qquad \psi \Rightarrow \psi \ \ (\mathit{Axioms})}{\varphi, \varphi \to \psi \Rightarrow \psi}\,(\to\Rightarrow)}{[a]\varphi, [a](\varphi \to \psi) \Rightarrow [a]\psi}\,([a] \Rightarrow [a])}{[a](\varphi \to \psi) \Rightarrow [a]\varphi \to [a]\psi}\,(\Rightarrow\to)}{\Rightarrow [a](\varphi \to \psi) \to ([a]\varphi \to [a]\psi)}\,(\Rightarrow\to)$$

Axiom $[a]\neg$, the $\to$ direction (*not* an Intuitionistic derivation)

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \ \ (\mathit{Axiom})}{\varphi \Rightarrow \varphi, \bot}\,(\Rightarrow \ \mathit{Weak})}{\Rightarrow \varphi, (\varphi \to \bot)}\,(\Rightarrow\to)}{\Rightarrow [a]\varphi, [a](\varphi \to \bot)}\,([a] \Rightarrow [a]) \qquad \bot \Rightarrow \ \ (\bot \Rightarrow)}{[a]\varphi \to \bot \Rightarrow [a](\varphi \to \bot)}\,(\to\Rightarrow)}{\Rightarrow ([a]\varphi \to \bot) \to [a](\varphi \to \bot)}\,(\Rightarrow\to)$$

Axiom $[a]\neg$, the $\leftarrow$ direction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi\ (Axiom) \qquad \perp\ \Rightarrow\quad (\perp\Rightarrow)}{(\varphi \to \perp),\ \varphi \Rightarrow}\ (\to\Rightarrow)}{[a](\varphi \to \perp),\ [a]\varphi \Rightarrow}\ ([a] \Rightarrow [a])}{[a](\varphi \to \perp),\ [a]\varphi \Rightarrow \perp}\ (\Rightarrow Weak)}{[a](\varphi \to \perp)\ \Rightarrow [a]\varphi \to \perp}\ (\Rightarrow\to)}{\Rightarrow [a](\varphi \to \perp) \to ([a]\varphi \to \perp)}\ (\Rightarrow\to)$$

■

We conclude this section with some rules admissible in the cut-free sequent calculus $\mathbf{S4F}_{G^-}$, which are used in the proof of completeness in Section 6.

Proposition 4.3 *Let Γ and Δ be multisets of formulas of $\mathcal{L}_{\Box a}$, let φ, ψ be formulas of $\mathcal{L}_{\Box a}$, and let $k \in \mathbb{N}$. The following rules are admissible in the cut-free sequent calculus $\mathbf{S4F}_{G^-}$.*

$$([a]^k \to\Rightarrow): \quad \dfrac{\Gamma \Rightarrow [a]^k\varphi \quad [a]^k\psi \Rightarrow \Delta}{[a]^k(\varphi \to \psi), \Gamma \Rightarrow \Delta}$$

$$(\Rightarrow [a]^k \to): \quad \dfrac{[a]^k\varphi, \Gamma \Rightarrow \Delta, [a]^k\psi}{\Gamma \Rightarrow \Delta, [a]^k(\varphi \to \psi)}$$

$$([a]^k\Box \Rightarrow): \quad \dfrac{[a]^k\varphi, \Gamma \Rightarrow \Delta}{[a]^k\Box\varphi, \Gamma \Rightarrow \Delta}$$

Proof. An argument is a pretty standard one for cut-free derivations. A straightforward strategy in each case should be to first apply the appropriate connective/modality rule, $(\to\Rightarrow)$, $(\Rightarrow\to)$ and $(\Box \Rightarrow)$, respectively, then deal with the $[a]^k$ prefix. We leave this to a reader as a routine exercise. ■

5 Kripke Semantics

Definition 5.1 *A* Kripke frame *for $\mathcal{L}_{\Box a}$ is a triple $\mathcal{K} = (W, R, F)$, where*

- *$W \neq \emptyset$ is a set of "worlds";*
- *$R \subseteq W \times W$ is a reflexive and transitive binary relation on W; and*
- *$F : W \to W$ is a total function on W.*

A Kripke frame $\mathcal{K} = (W, R, F)$ is called finite *iff W is a finite set.*

By standard arguments, reflexive and transitive binary relations capture precisely the **S4** $\Box$ modality. As in [Lem77], §4, pp. 60-61, a total function $F : W \to W$ is used to interpret the $[a]$ modality. If one prefers to interpret modalities with a binary relation on W, take $Q = graph(F)$. Then as a binary relation, Q is both "total" and "functional", i.e. for all $w \in W$, there *exists a unique* $v \in W$ such that $(w, v) \in Q$. The "totality" or "serial" condition: every $w \in W$ has *at least* one Q-successor, is characteristic for the deontic scheme:

$$[a]\mathbf{D} : \quad [a]\varphi \to \langle a\rangle\varphi$$

The converse scheme:

$$[a]\mathbf{D}_c : \quad \langle a\rangle\varphi \to [a]\varphi$$

is characterized by the "functionality" or "determinism" condition: every $w \in W$ has *at most* one Q-successor.

Definition 5.2 *A* valuation *for a Kripke frame $\mathcal{K} = (W, R, F)$ is a map $\eta : W \to \mathcal{P}(AP)$ assigning a set of atomic propositions $\eta(w) \subseteq AP$ to each world $w \in W$. Each such valuation for $\mathcal{K}$ determines a forcing relation $\Vdash_\eta^{\mathcal{K}} = \Vdash_\eta \subseteq W \times AP$ defined by*

$$w \Vdash_\eta p \quad \text{iff} \quad p \in \eta(w)$$

which uniquely extends a forcing relation $\Vdash_\eta \subseteq W \times \mathcal{L}_{\Box a}$ (with the same name) on all formulas of $\mathcal{L}_{\Box a}$, by the following clauses:

(i) $w \Vdash_\eta \neg\varphi$ *iff* $w \nVdash_\eta \varphi$;

(ii) $w \Vdash_\eta \varphi \to \psi$ *iff* $w \nVdash_\eta \varphi$ *or* $w \Vdash_\eta \psi$;

(iii) $w \Vdash_\eta \Box\varphi$ *iff for all* $v \in W$, *if* $(w, v) \in R$ *then* $v \Vdash_\eta \varphi$;

(iv) $w \Vdash_\eta [a]\varphi$ *iff* $F(w) \Vdash_\eta \varphi$.

for all $w \in W$, and all $\varphi, \psi \in \mathcal{L}_{\Box a}$.

If $Q = graph(F)$, then by the total functionality of Q, this last clause is equivalent to

$$w \Vdash_\eta [a]\varphi \quad \text{iff} \quad \text{for all } v \in W, \text{ if } (w, v) \in Q \text{ then } v \Vdash_\eta \varphi.$$

Definition 5.3 *A* Kripke model *for $\mathcal{L}_{\Box a}$ is a pair $(\mathcal{K}, \eta)$, where $\mathcal{K} = (W, R, F)$ is a frame for $\mathcal{L}_{\Box a}$ and $\eta : W \to \mathcal{P}(AP)$ is a valuation for $\mathcal{K}$.*

Definition 5.4 *Let φ be a propositional formula of $\mathcal{L}_{\Box a}$.*

- *φ is* satisfied *(or* forced*) at a world $w \in W$ in a Kripke model $(\mathcal{K}, \eta)$ iff $w \Vdash_\eta^{\mathcal{K}} \varphi$;*

- φ *is* true *in a Kripke model* $(\mathcal{K}, \eta)$, *written* $(\mathcal{K}, \eta) \Vdash \varphi$, *iff for* all *worlds* $w \in W$, *we have* $w \Vdash^{\mathcal{K}}_{\eta} \varphi$;

- φ *is* valid *in a frame* $\mathcal{K}$, *written* $\mathcal{K} \Vdash \varphi$, *iff for* all *valuations* $\eta : W \to \mathcal{P}(AP)$ *for* $\mathcal{K}$, *we have* $(\mathcal{K}, \eta) \Vdash \varphi$;

- φ *is* Kripke valid *iff for all frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$, $\mathcal{K} \Vdash \varphi$.

Proposition 5.5 Kripke Soundness of **S4F** Hilbert-style proof system
For all formulas φ *of* $\mathcal{L}_{\Box a}$, *if* $\mathbf{S4F} \vdash_H \varphi$ *then* φ *is Kripke valid.*

Proof. The required verification is that each of the axioms of $\mathbf{S4F}_H$ are Kripke valid, and that the inference rules of $\mathbf{S4F}_H$ preserve Kripke validity. For the axioms **CP** of classical propositional logic and for *modus ponens*, this is trivial. The verification for the **S4** axioms **K**, **T** and **4**, and the $\Box$-necessitation rule follow the standard proof of soundness of the class of transitive and reflexive frames for **S4**; see, for example, [HC96], pp.56-57. For the $[a]$-necessitation rule, suppose φ is Kripke valid, let $\mathcal{K} = (W, R, F)$ be a frame for $\mathcal{L}_{\Box a}$, and let η be a valuation for $\mathcal{K}$. Since φ is Kripke valid and $F(w) \in W$ since F is total, we have $F(w) \Vdash_\eta \varphi$. Hence $w \Vdash_\eta \varphi$. Hence $[a]\varphi$ is also Kripke valid. The verification of the validity of the $[a]$**K** and $[a]$**F** axioms is also straightforward, taking as a starting point the fact that for any formula φ and any $w \in W$, either $F(w) \Vdash_\eta \varphi$ or $F(w) \nVdash_\eta \varphi$, and then crunching through the definitions of forcing for $\neg$, $\to$ and $[a]$. ■

Proposition 5.6 *For all formulas* φ *of* $\mathcal{L}_{\Box a}$,
if $\mathfrak{T} \models \varphi$ *for all topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$,
then $\mathcal{K} \Vdash \varphi$ *for all Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$.

Proof. Given a Kripke frame $\mathcal{K} = (W, R, F)$ be a for $\mathcal{L}_{\Box a}$, define $\mathcal{T}_R$ to be the topology on W which has as a basis the collection of all sets

$$B_w = \{v \in W \mid (w, v) \in R\}^3$$

So B_w is the set of all R-successors of w. Note that $w \in B_w$ (by the reflexivity of R) and $v \in B_w$ implies $B_v \subseteq B_w$ (by the transitivity of R), so $B_w = \bigcup_{v \in B_w} B_v$. It is readily verified that for any set $A \subseteq W$, we have:

$$\begin{aligned} int_{\mathcal{T}_R}(A) &= \{w \in W \mid B_w \subseteq A\} \\ &= \{w \in W \mid \text{for all } v \in W, \text{ if } (w, v) \in R \text{ then } v \in A\} \end{aligned}$$

[3] The topology $\mathcal{T}_R$ is variously known as the "cone topology" (generated from R-cones B_w) and the "Alexandroff topology" (from [Ale56], where R is a partial order). Grzegorczyk uses an equivalent topology in [Grz67].

In particular, an open set $U \in \mathcal{T}_R$ is a neighborhood of w iff $B_w \subseteq U$.

Since $F : W \to W$ is a total function, the induced structure $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ is a topological structure for $\mathcal{L}_{\Box a}$. Given a valuation $\eta : W \to \mathcal{P}(AP)$ for $\mathcal{K}$, define its dual valuation $\xi_\eta : AP \to \mathcal{P}(W)$ for $\mathfrak{T}_{\mathcal{K}}$ by:

$$w \in \xi_\eta(p) \quad \text{iff} \quad p \in \eta(w)$$

for all $p \in AP$ and $w \in W$. A simple induction on formulas establishes that for all $\varphi \in \mathcal{L}_{\Box a}$ and all $w \in W$,

$$w \in \|\varphi\|_{\xi_\eta} \quad \text{iff} \quad w \Vdash_\eta \varphi$$

Hence

$$(\mathfrak{T}_{\mathcal{K}}, \xi_\eta) \models \varphi \quad \text{iff} \quad (\mathcal{K}, \eta) \Vdash \varphi$$

and the result follows. ■

6 Kripke Completeness for S4F

Our task in this section is Kripke completeness for **S4F**, together with the finite model property, and a semantic proof of cut-elimination. We prove that for all sequents $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$, if $\Gamma_0 \Rightarrow \Delta_0$ does not have a cut-free proof in $\mathbf{S4F}_{G-}$, then there is a *finite* Kripke model $(\mathcal{K}, \eta)$ for $\mathcal{L}_{\Box a}$ such that at a world w_0 of $\mathcal{K}$, we have $w_0 \nVdash^{\mathcal{K}}_\eta \bigwedge \Gamma_0 \to \bigvee \Delta_0$, i.e. $w_0 \Vdash^{\mathcal{K}}_\eta \varphi$ for each formula φ occurring in the antecedent Γ_0, and $w_0 \nVdash^{\mathcal{K}}_\eta \psi$ for each formula ψ occurring in the succedent Δ_0.

The fundamental notion is that of a *saturated* sequent. A sequent $\Gamma \Rightarrow \Delta$ in the language $\mathcal{L}_{\Box a}$ (in fact, in the language $\mathcal{L}_\Box$) is called **S4** *saturated* iff each the following conditions hold:

(1.) if $\varphi \to \psi \in \Gamma$ then either $\psi \in \Gamma$ or $\varphi \in \Delta$;

(2.) if $\varphi \to \psi \in \Delta$ then both $\varphi \in \Gamma$ and $\psi \in \Delta$;

(3.) if $\Box\varphi \in \Gamma$ then $\varphi \in \Gamma$,

for all $\varphi, \psi \in \mathcal{L}_{\Box a}(\mathcal{L}_\Box)$. Trivially, the empty sequent, $\emptyset \Rightarrow \emptyset$, is **S4** saturated.

Variants of the notion of saturation for sequents are found throughout the modal and non-classical logic literature; see, for example, [AS93], [Av84]. This notion is intimately related with the notion of a set of signed formulas as a *consistency property* in [Fi83]. The saturation algorithm below is modelled on that of [AS93]. Here, we strengthen the notion of saturation to deal with the $[a]$ operator.

Definition 6.1

A sequent $\Gamma \Rightarrow \Delta$ *of* $\mathcal{L}_{\Box a}$ *is called* **S4F** saturated *iff each the following conditions hold:*

(1.) *if* $[a]^k(\varphi \to \psi) \in \Gamma$ *then either* $[a]^k\psi \in \Gamma$ *or* $[a]^k\varphi \in \Delta$*;*

(2.) *if* $[a]^k(\varphi \to \psi) \in \Delta$ *then both* $[a]^k\varphi \in \Gamma$ *and* $[a]^k\psi \in \Delta$*;*

(3.) *if* $[a]^k\Box\varphi \in \Gamma$ *then* $[a]^k\varphi \in \Gamma$*;*

for all $\varphi, \psi \in \mathcal{L}_{\Box a}$ *and* $k \in \mathbb{N}$*.*

It is immediate that if $\Gamma \Rightarrow \Delta$ is **S4F** saturated, then $\Gamma \Rightarrow \Delta$ is **S4** saturated, since **S4** saturation is just the case of $k = 0$ in each of conditions (1.), (2.) and (3.). In the stronger notion of **S4F** saturation, we require that subformulas behave "appropriately" with respect to iterated $[a]^k$'s. Note that each of the conditions is reflected in an admissible rule for $\mathbf{S4F}_{G-}$, as given in Proposition 4.3.

As a technical point, $SubForm(\Gamma_0 \cup \Delta_0)$ should be treated as a multiset: for each formula φ occurring in the multiset $\Gamma_0 \cup \Delta_0$, the multiset of all subformulas of φ is contained in $SubForm(\Gamma_0 \cup \Delta_0)$. In particular, expressions of the form

$$\Gamma \cup \Delta \subseteq SubForm(\Gamma_0 \cup \Delta_0) \tag{1}$$

are to be read as multiset inclusion. Given a sequent $\Gamma_0 \Rightarrow \Delta_0$, there are only finitely many sequents $(\Gamma \Rightarrow \Delta)$ of $\mathcal{L}_{\Box a}$ such that the equation above is satisfied.

Lemma 6.2 S4F Saturation
For each sequent $\Gamma_0 \Rightarrow \Delta_0$ *of* $\mathcal{L}_{\Box a}$*,*
if $\mathbf{S4F} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$*, then there is an* **S4F** *saturated sequent* $\Gamma \Rightarrow \Delta$ *such that*

(a) $\Gamma_0 \subseteq \Gamma \subseteq SubForm(\Gamma_0 \cup \Delta_0)$*;*

(b) $\Delta_0 \subseteq \Delta \subseteq SubForm(\Gamma_0 \cup \Delta_0)$*;*

(c) $\mathbf{S4F} \nvdash_{G-} \Gamma \Rightarrow \Delta$.

Moreover, by determinizing the algorithm which produces such a saturated sequent from input $\Gamma_0 \Rightarrow \Delta_0$*, we may take the output* $\Gamma \Rightarrow \Delta$ *to be unique, and denote it* $Sat(\Gamma_0 \Rightarrow \Delta_0)$*, the* **S4F** *saturation of* $\Gamma_0 \Rightarrow \Delta_0$*.*

Proof. We expand on the saturation algorithm of [AS93], taking care to eliminate any non-determinism. Given as input a sequent $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$ such that $\mathbf{S4F} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$, we construct a finite tree $\mathbf{T}(\Gamma_0 \Rightarrow \Delta_0)$ labelled with sequents of $\mathcal{L}_{\Box a}$ such that:

(i) the root node of $\mathbf{T}(\Gamma_0 \Rightarrow \Delta_0)$ is labelled by $\Gamma_0 \Rightarrow \Delta_0$;

(ii) all sequents $\Gamma \Rightarrow \Delta$ labelling nodes in $\mathbf{T}(\Gamma_0 \Rightarrow \Delta_0)$ satisfy:

(a) $\Gamma_0 \subseteq \Gamma \subseteq SubForm(\Gamma_0 \cup \Delta_0)$;

(b) $\Delta_0 \subseteq \Delta \subseteq SubForm(\Gamma_0 \cup \Delta_0)$.

The algorithm requires a sub-routine **Marking**, which is a book-keeping device for keeping track of which formulas have been dealt with or are yet to be dealt with.

Marking($\Gamma \Rightarrow \Delta$): Mark each occurrence of a formula in $\Gamma \cup \Delta$ with either a "0" (yet to be dealt with) or a "1" (dealt with) as follows:

- Each occurrence of a propositional variable or $\bot$ in $\Gamma \cup \Delta$ is marked "1".
- For each occurrence of a formula $[a]^k(\varphi \to \psi)$ in Γ, if there is no occurrence of $[a]^k\psi$ in Γ and there is also no occurrence of $[a]^k\varphi$ in Δ, then mark the $[a]^k(\varphi \to \psi)$ with "0"; otherwise, mark it with "1".
- For each occurrence of a formula $[a]^k(\varphi \to \psi)$ in Δ, if there is an occurrence of $[a]^k\varphi$ in Γ and there is also an occurrence of $[a]^k\psi$ in Δ, then mark the $[a]^k(\varphi \to \psi)$ with "1"; otherwise, mark it with "0".
- For each occurrence of a formula $[a]^k\Box\varphi$ in Γ, if there is no occurrence of $[a]^k\varphi$ in Γ then mark the $[a]^k\Box\varphi$ with "0"; otherwise, mark it with "1".
- All remaining occurrences of formulas in $\Gamma \cup \Delta$ are marked "1".

Initialize: The *current node* is the root node labelled $\Gamma_0 \Rightarrow \Delta_0$. Run the sub-routine **Marking($\Gamma_0 \Rightarrow \Delta_0$)**.

Repeat with each *current node*, labelled $\Gamma \Rightarrow \Delta$:

0. Axiom Test: Check if $\Gamma \cap \Delta \neq \emptyset$, or if $\bot \in \Gamma$.

If either of these tests are satisfied, put a check mark "✓" next to the current node then backtrack up the tree to the first ancestor of the current node that is a branching node and has a child node without a check mark, then select the check-less (always the right) child as the new *current node*. [If all children of all branching ancestors of the current node are checked, then the tree $\mathbf{T}(\Gamma_0 \Rightarrow \Delta_0)$ can be easily transformed into a cut-free proof in $\mathbf{S4F}_G^-$ of $\Gamma_0 \Rightarrow \Delta_0$ (using only (*Axiom*), ($\bot \Rightarrow$), the admissible rules ($[a]^k \to\Rightarrow$), ($\Rightarrow [a]^k \to$) and ($[a]^k\Box \Rightarrow$), plus the weakening and contraction rules), which contradicts the assumption that $\mathbf{S4F} \nvdash_{G^-} \Gamma_0 \Rightarrow \Delta_0$.]

If $\Gamma \cap \Delta = \emptyset$ and $\bot \notin \Gamma$, proceed to **1.** working with the current node.

1. Antecedent $[a]^k \to$: If Γ contains an occurrence of a formula $[a]^k(\varphi \to \psi)$ marked "0", put a check mark "✓" next to the current node, then create two child nodes:

$$\dfrac{\Gamma \Rightarrow \Delta, [a]^k\varphi \qquad \Gamma, [a]^k\psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta \quad \checkmark}$$

labelled $\Gamma \Rightarrow \Delta, [a]^k\varphi$ and $\Gamma, [a]^k\psi \Rightarrow \Delta$, respectively. Run the marking sub-routine on both child nodes: **Marking(**$\Gamma \Rightarrow \Delta, [a]^k\varphi$**)**, and **Marking(**$\Gamma, [a]^k\psi \Rightarrow \Delta$**)**. Then select the left child node, labelled $\Gamma \Rightarrow \Delta, [a]^k\varphi$, as the new *current node*.

If Γ contains no occurrences of any formula $[a]^k(\varphi \to \psi)$ marked "0", proceed to **2.** working with the current node.

2. Succedent $[a]^k \to$: If Δ contains an occurrence of a formula $[a]^k(\varphi \to \psi)$ marked "0", put a check mark "✓" next to the current node, then create one child node:

$$\begin{array}{c} \Gamma, [a]^k\varphi \Rightarrow \Delta, [a]^k\psi \\ | \\ \Gamma \Rightarrow \Delta \quad \checkmark \end{array}$$

labelled $\Gamma, [a]^k\varphi \Rightarrow \Delta, [a]^k\psi$. Run the sub-routine **Marking(**$\Gamma, [a]^k\varphi \Rightarrow \Delta, [a]^k\psi$**)**. Select the child node labelled $\Gamma, [a]^k\varphi \Rightarrow \Delta, [a]^k\psi$ as the new *current node*.

If Δ contains no occurrences of any formula $[a]^k(\varphi \to \psi)$ marked "0", proceed to **3.** working with the current node.

3. Antecedent $[a]^k\Box$: If Γ contains an occurrence of a formula $[a]^k\Box\varphi$ marked "0", put a check mark "✓" next to the current node, then create one child node:

$$\begin{array}{c} \Gamma, [a]^k\varphi \Rightarrow \Delta \\ | \\ \Gamma \Rightarrow \Delta \quad \checkmark \end{array}$$

labelled $\Gamma, [a]^k\varphi \Rightarrow \Delta$. Run the sub-routine **Marking(**$\Gamma, [a]^k\varphi \Rightarrow \Delta$**)**. Select the child node labelled $\Gamma, [a]^k\varphi \Rightarrow \Delta$ as the new *current node*.

If Γ contains no occurrences of any formula $[a]^k\Box\varphi$ marked "0", then proceed to **4.**

4. Terminate and return the label of the current node, $\Gamma \Rightarrow \Delta$ (which does NOT have a check mark "✓") as *the* saturation of $\Gamma_0 \Rightarrow \Delta_0$, i.e. $Sat(\Gamma_0 \Rightarrow \Delta_0) = \Gamma \Rightarrow \Delta$

The saturation algorithm must terminate because $SubForm(\Gamma_0 \cup \Delta_0)$ is finite and there is at most two branches at each step.

It is immediate from the construction that if $\Gamma \Rightarrow \Delta = Sat(\Gamma_0 \Rightarrow \Delta_0)$ then

(a) $\Gamma_0 \subseteq \Gamma \subseteq SubForm(\Gamma_0 \cup \Delta_0)$ and

(b) $\Delta_0 \subseteq \Delta \subseteq SubForm(\Gamma_0 \cup \Delta_0)$

hold. To see that

(c) **S4F** $\nvdash_{G-} \Gamma \Rightarrow \Delta$

also holds, observe that if $\Gamma \Rightarrow \Delta$ had a cut-free proof in $\mathbf{S4F}_{G-}$, then by saturation, we would have $\Gamma \cap \Delta \neq \emptyset$ or $\bot \in \Gamma$; from such axiom sequents, we could reverse the steps in the saturation process to construct a cut-free proof of $\Gamma_0 \Rightarrow \Delta_0$, contradicting the assumption that $\mathbf{S4F} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$. ■

As a corollary of the proof (the **Marking** sub-routine), we have that for sequents $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\Box a}$ with $\mathbf{S4F} \nvdash_{G-} \Gamma \Rightarrow \Delta$,

$$\Gamma \Rightarrow \Delta \text{ is saturated iff } Sat(\Gamma \Rightarrow \Delta) = \Gamma \Rightarrow \Delta$$

To deal with formulas having $[a]$ as the main operator/connective, we define an operation on sequents called "*Strip*".

Definition 6.3 *For any sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\Box a}$, define*

$$Strip(\Gamma \Rightarrow \Delta) \ \hat{=} \ \{\{\varphi \mid [a]\varphi \in \Gamma\}\} \Rightarrow \{\{\psi \mid [a]\psi \in \Delta\}\}$$

where the double braces $\{\{...\}\}$ denote multi-set *formation.*

So if $Strip(\Gamma \Rightarrow \Delta) = (\Gamma' \Rightarrow \Delta')$, then for each occurrence of a formula $[a]\varphi$ in Γ, there is a corresponding occurrence of φ in Γ', and likewise, for each occurrence of a formula $[a]\psi$ in Δ, there is a corresponding occurrence of ψ in Δ', and these are the *only* formulas occurring in Γ' and Δ' respectively. In particular, all formulas in $\Gamma \cup \Delta$ that do *not* have $[a]$ as the main operator/connective are erased completely by the *Strip* operator. Thus if there are no occurrences of formulas of the form $[a]\varphi$ in $\Gamma \cup \Delta$, then $Strip(\Gamma \Rightarrow \Delta) = (\emptyset \Rightarrow \emptyset)$, the empty sequent.

Lemma 6.4 *For all sequents $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\Box a}$, if $Strip(\Gamma \Rightarrow \Delta) = \Gamma' \Rightarrow \Delta'$, then for all $\varphi \in \mathcal{L}_{\Box a}$,*

(i) $\Gamma' \subseteq SubForm(\Gamma)$ *and* $\Delta' \subseteq SubForm(\Delta)$*;*

(ii) $[a]\varphi \in \Gamma$ *iff* $\varphi \in \Gamma'$*;*

(iii) $[a]\varphi \in \Delta$ *iff* $\varphi \in \Delta'$*;*

(iv) *if* $\mathbf{S4F} \nvdash_{G-} \Gamma \Rightarrow \Delta$, *then* $\mathbf{S4F} \nvdash_{G-} \Gamma' \Rightarrow \Delta'$*;*

(v) *if* $\Gamma \Rightarrow \Delta$ *is* **S4F** *saturated, then* $\Gamma' \Rightarrow \Delta'$ *is also* **S4F** *saturated.*

Proof. Properties (i), (ii) and (iii) are immediate from the definition of *Strip*. For (iv), suppose $\mathbf{S4F} \nvdash_{G-} \Gamma \Rightarrow \Delta$, but $\mathbf{S4F} \vdash_{G-} \Gamma' \Rightarrow \Delta'$. Then from a cut-free proof of $\Gamma' \Rightarrow \Delta'$, one can construct a cut-free proof of $\Gamma \Rightarrow \Delta$ using the $([a] \Rightarrow [a])$ rule followed by left (respectively, right) weakening of all the formulas in Γ (respectively, Δ) that do not have $[a]$ as the main operator/connective. For (v), suppose $\Gamma \Rightarrow \Delta$ is **S4F** saturated, and consider $\Gamma' \Rightarrow \Delta'$. Then for clause (1.) of **S4F** saturation,

$$\begin{array}{lll} & [a]^k(\varphi \to \psi) \in \Gamma' & \\ \Leftrightarrow & [a]^{k+1}(\varphi \to \psi) \in \Gamma & \text{by (ii)} \\ \Rightarrow & [a]^{k+1}\psi \in \Gamma \ \text{ or } [a]^{k+1}\varphi \in \Delta & \text{by } \mathbf{S4F} \text{ saturation of } \Gamma \Rightarrow \Delta \\ \Leftrightarrow & [a]^k\psi \in \Gamma' \ \text{ or } [a]^k\varphi \in \Delta' & \text{by (ii) and (iii)} \end{array}$$

The verification for clauses (2.) and (3.) proceeds similarly. ■

As is suggested by the name, the *Strip* function "strips off" outermost $[a]$'s, thus reducing the complexity of the sequent with respect to the nesting of $[a]$'s. The following definition makes this more precise.

Definition 6.5 *For formulas φ of $\mathcal{L}_{\Box a}$, define $[a]\text{rank}(\varphi)$ in the obvious way:*

$$\begin{array}{rll} [a]\text{rank}(q) & = 0 & \text{for } q \in AP \cup \{\bot\} \\ [a]\text{rank}(\varphi \to \psi) & = \max\{[a]\text{rank}(\varphi), [a]\text{rank}(\psi)\} & \\ [a]\text{rank}(\Box\varphi) & = [a]\text{rank}(\varphi) & \\ [a]\text{rank}([a]\varphi) & = [a]\text{rank}(\varphi) + 1 & \end{array}$$

And for a sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\Box a}$, define

$$[a]\text{rank}(\Gamma \Rightarrow \Delta) = \max\{[a]\text{rank}(\varphi) \mid \varphi \text{ in } \Gamma \cup \Delta\}$$

Lemma 6.6 *For any sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\Box a}$,*

(a) *if $\Gamma \cup \Delta$ contains at least one formula of the form $[a]\varphi$, then*

$$[a]\text{rank}\left(Strip(\Gamma \Rightarrow \Delta)\right) \leq [a]\text{rank}(\Gamma \Rightarrow \Delta) - 1$$

and otherwise

$$[a]\text{rank}\left(Strip(\Gamma \Rightarrow \Delta)\right) = 0$$

(b) *if $[a]\text{rank}(\Gamma \Rightarrow \Delta) = m$ then $Strip^{m+1}(\Gamma \Rightarrow \Delta) = (\emptyset \Rightarrow \emptyset)$.*

Proof. Immediate from Definition 6.5. ■

Definition 6.7 *For each sequent $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$, define Sub-*$\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$ *to be the set of all sequents $\Gamma \Rightarrow \Delta$ in $\mathcal{L}_{\Box a}$ satisfying the three properties:*

- $\Gamma \Rightarrow \Delta$ *is* **S4F** *saturated;*
- $\Gamma \cup \Delta \subseteq SubForm(\Gamma_0 \cup \Delta_0)$*;*
- $\mathbf{S4F} \nvdash_{G-} \Gamma \Rightarrow \Delta$.

It is immediate from the second property that Sub-$\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$ is finite; it is also non-empty since it always contains the empty sequent, $(\emptyset \Rightarrow \emptyset)$. Note that if $\mathbf{S4F} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$, then $Sat(\Gamma_0 \Rightarrow \Delta_0) = \Gamma_1 \Rightarrow \Delta_1$ is in Sub-$\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$.

Definition 6.8 *For each sequent $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$, we define a Kripke frame $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)} = (W, R, F)$ for $\Gamma_0 \Rightarrow \Delta_0$ as follows:*

- $W = Sub$-$\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$*;*
- $\big((\Gamma \Rightarrow \Delta), (\Gamma' \Rightarrow \Delta')\big) \in R$ *iff* $\Box\varphi \in \Gamma$ *implies* $\Box\varphi \in \Gamma'$
- $F = Strip$

The Kripke frame $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ is called the **S4F** saturation frame *for $\Gamma_0 \Rightarrow \Delta_0$.*
Define the canonical valuation $\eta : W \to \mathcal{P}(\mathbf{PV})$ for $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ by

$$p \in \eta(\Gamma \Rightarrow \Delta) \quad \text{iff} \quad p \in \Gamma$$

It is readily verified that the **S4F** saturation frame $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ *is* a Kripke frame for $\mathcal{L}_{\Box a}$. The reflexivity and transitivity of R follow from the corresponding properties of implication, and by Lemma 6.4, $F = Strip : W \to W$ is a total function on $W = Sub$-$\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$.

Lemma 6.9 Main Semantic Lemma for **S4F**

Let $\Gamma_0 \Rightarrow \Delta_0$ be any sequent in $\mathcal{L}_{\Box a}$, let $\mathcal{K} = \mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ be the **S4F** *saturation frame for $\Gamma_0 \Rightarrow \Delta_0$, and let η be the canonical valuation for $\mathcal{K}$ as in Definition 6.8.*

Then for all $(\Gamma \Rightarrow \Delta) \in W$ and for all formulas φ in $\mathcal{L}_{\Box a}$, we have:

$$\begin{array}{lll} \varphi \in \Gamma & \text{implies} & (\Gamma \Rightarrow \Delta) \Vdash_\eta \varphi \\ \varphi \in \Delta & \text{implies} & (\Gamma \Rightarrow \Delta) \nVdash_\eta \varphi \end{array}$$

Proof. We proceed by induction on the complexity of formulas φ in $\mathcal{L}_{\Box a}$.

Fix $(\Gamma \Rightarrow \Delta) \in W = Sub\text{-}\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$. For propositional variables $p \in \mathbf{PV}$, $p \in \Gamma$ implies $(\Gamma \Rightarrow \Delta) \Vdash_\eta p$, directly from the definition of atomic forcing, and $p \in \Delta$ implies $p \notin \Gamma$ since $\mathbf{S4F} \nvdash_{G^-} \Gamma \Rightarrow \Delta$, hence $(\Gamma \Rightarrow \Delta) \nVdash_\eta p$ from the definition of atomic forcing. For the constant $\bot$, the condition $\bot \in \Gamma$ is impossible, since $\mathbf{S4F} \nvdash_{G^-} \Gamma \Rightarrow \Delta$, and $(\Gamma \Rightarrow \Delta) \nVdash_\eta \bot$ by the definition of $\Vdash_\eta$for $\bot$, hence the result holds for $\bot$.

For $\rightarrow$, assume by induction that the result holds for φ and ψ, for all sequents in W. Fix $(\Gamma \Rightarrow \Delta) \in W$ and suppose $\varphi \rightarrow \psi \in \Gamma$. Then by the $\mathbf{S4F}$ saturation of $\Gamma \Rightarrow \Delta$ (clause (1.), $k = 0$), we have either $\psi \in \Gamma$ or $\varphi \in \Delta$. Hence by the induction hypothesis, $(\Gamma \Rightarrow \Delta) \Vdash_\eta \psi$ or $(\Gamma \Rightarrow \Delta) \nVdash_\eta \varphi$. Hence $(\Gamma \Rightarrow \Delta) \Vdash_\eta \varphi \rightarrow \psi$. For the succeedent, suppose $\varphi \rightarrow \psi \in \Delta$. Then by the $\mathbf{S4F}$ saturation of $\Gamma \Rightarrow \Delta$ (clause (2.), $k = 0$), we have $\varphi \in \Gamma$ and $\psi \in \Delta$. Hence by the induction hypothesis, $(\Gamma \Rightarrow \Delta) \Vdash_\eta \varphi$ and $(\Gamma \Rightarrow \Delta) \nVdash_\eta \psi$. Hence $(\Gamma \Rightarrow \Delta) \nVdash_\eta \varphi \rightarrow \psi$.

For $\Box$ in the antecedent, assume by induction that the result holds for φ, for all sequents in W. Fix $(\Gamma \Rightarrow \Delta) \in W$ and suppose $\Box\varphi \in \Gamma$. Now let $(\Gamma' \Rightarrow \Delta') \in W$ be any sequent such that $\big((\Gamma \Rightarrow \Delta), (\Gamma' \Rightarrow \Delta')\big) \in R$. Then $\Box\varphi \in \Gamma'$, by the definition of R, and then by the $\mathbf{S4F}$ saturation of $\Gamma' \Rightarrow \Delta'$ (clause (3.), $k = 0$), we have $\varphi \in \Gamma'$. Hence by the induction hypothesis, $(\Gamma' \Rightarrow \Delta') \Vdash_\eta \varphi$. Thus by the definition of $\Vdash_\eta$for $\Box$, we have $(\Gamma \Rightarrow \Delta) \Vdash_\eta \Box\varphi$.

For $\Box$ in the succedent, assume by induction that the result holds for φ, for all sequents in W. Fix $(\Gamma \Rightarrow \Delta) \in W$ and suppose $\Box\varphi \in \Delta$. Let $\Box\psi_1, ..., \Box\psi_n$ be a list of all occurrences of formulas in Γ which have $\Box$ as their main connective/operator. Let $\Gamma' \Rightarrow \Delta'$ be the sequent $\Box\psi_1, ..., \Box\psi_n \Rightarrow \varphi$. Then $\mathbf{S4F} \nvdash_{G^-} \Gamma' \Rightarrow \Delta'$, for otherwise, from a cut-free proof of $\Gamma' \Rightarrow \Delta'$, we could construct a cut-free proof of $\Gamma \Rightarrow \Delta$ using the rule $(\Rightarrow \Box)$ plus left and right weakening, thus contradicting $\mathbf{S4F} \nvdash_{G^-} \Gamma \Rightarrow \Delta$. Now let $(\Gamma'' \Rightarrow \Delta'') = Sat(\Gamma' \Rightarrow \Delta')$. Then from Lemma 6.2,

- $(\Gamma'' \Rightarrow \Delta'')$ is $\mathbf{S4F}$ saturated;
- $\Gamma'' \subseteq SubForm(\Gamma') \subseteq SubForm(\Gamma_0 \cup \Delta_0)$, and
  $\Delta'' \subseteq SubForm(\Delta') \subseteq SubForm(\Gamma_0 \cup \Delta_0)$; and
- $\mathbf{S4F} \nvdash_{G^-} \Gamma'' \Rightarrow \Delta''$.

Hence $(\Gamma'' \Rightarrow \Delta'') \in W = Sub\text{-}\mathbf{S4F}(\Gamma_0 \Rightarrow \Delta_0)$. Moreover, $\Big((\Gamma \Rightarrow \Delta), (\Gamma'' \Rightarrow \Delta'')\Big) \in R$, since $\Box\psi_i \in \Gamma$ implies $\Box\psi_i \in \Gamma''$. Now $\varphi \in \Delta''$, hence by the induction hypothesis, $(\Gamma'' \Rightarrow \Delta'') \nVdash_\eta \varphi$. Then by the definition of $\Vdash_\eta$for $\Box$, we have $(\Gamma \Rightarrow \Delta) \nVdash_\eta \Box\varphi$.

Finally for $[a]$, assume by induction that the result holds for φ, for all sequents in W. Fix $(\Gamma \Rightarrow \Delta) \in W$, and let $\Gamma' \Rightarrow \Delta' = F(\Gamma \Rightarrow \Delta) = Strip(\Gamma \Rightarrow \Delta)$. Then $[a]\varphi \in \Gamma$ implies $\varphi \in \Gamma'$, by Lemma 6.4, hence by the induction hypothesis, $(\Gamma' \Rightarrow \Delta') \Vdash_\eta \varphi$. Then by the definition of $\Vdash_\eta$for $[a]$, we have $(\Gamma \Rightarrow \Delta) \Vdash_\eta [a]\varphi$. Symmetrically, for the succeedent,

$[a]\varphi \in \Delta$ implies $\varphi \in \Delta'$, by Lemma 6.4, hence by the induction hypothesis, $(\Gamma' \Rightarrow \Delta') \nVdash_\eta \varphi$. Then by the definition of $\Vdash_\eta$for $[a]$, we have $(\Gamma \Rightarrow \Delta) \nVdash_\eta [a]\varphi$. ■

Theorem 6.10 Kripke completeness and finite model property for **S4F**

Let $\Gamma_0 \Rightarrow \Delta_0$ be any sequent in $\mathcal{L}_{\Box a}$.

If **S4F** $\nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$*, then there is a finite Kripke frame $\mathcal{K}$ and valuation η for $\mathcal{K}$ such that $(\mathcal{K}, \eta) \nVdash \bigwedge \Gamma_0 \to \bigvee \Delta_0$.*

Proof. Let $\mathcal{K} = \mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ be the **S4F** saturation frame for $\Gamma_0 \Rightarrow \Delta_0$, let η be the canonical valuation for $\mathcal{K}$, as in Definition 6.8, and let $(\Gamma_1 \Rightarrow \Delta_1) = Sat(\Gamma_0 \Rightarrow \Delta_0)$. If **S4F** $\nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$ then $(\Gamma_1 \Rightarrow \Delta_1) \in W$. Since $\Gamma_0 \subseteq \Gamma_1$ and $\Delta_0 \subseteq \Delta_1$, we have by Lemma 6.9,

$$\begin{array}{rl} & (\Gamma_1 \Rightarrow \Delta_1) \Vdash_\eta \varphi \ \text{ for all } \varphi \in \Gamma_0 \\ \text{and} & (\Gamma_1 \Rightarrow \Delta_1) \nVdash_\eta \psi \ \text{ for all } \psi \in \Delta_0 \end{array}$$

hence

$$(\mathcal{K}, \eta) \nVdash \bigwedge \Gamma_0 \to \bigvee \Delta_0$$

■

7 Consolidation Theorems for S4F

We consolidate the major results of previous sections.

Theorem 7.1 *For all multisets Γ, Δ of formulas of $\mathcal{L}_{\Box a}$, the following are equivalent:*

(1.) **S4F** $\vdash_{G-} \Gamma \Rightarrow \Delta$

(2.) **S4F** $\vdash_G \Gamma \Rightarrow \Delta$

(3.) **S4F** $\vdash_H \bigwedge \Gamma \to \bigvee \Delta$

(4.) $\mathfrak{T} \models \bigwedge \Gamma \to \bigvee \Delta$ *for all topological structures $\mathfrak{T}$ for $\mathcal{L}_{\Box a}$,*

(5.) $\mathcal{K} \Vdash \bigwedge \Gamma \to \bigvee \Delta$ *for all Kripke frames $\mathcal{K}$ for $\mathcal{L}_{\Box a}$,*

(6.) $\mathcal{K} \Vdash \bigwedge \Gamma \to \bigvee \Delta$ *for all finite Kripke frames $\mathcal{K}$ for $\mathcal{L}_{\Box a}$.*

Proof. (1.) $\Rightarrow$ (2.) is trivial. (2.) $\Leftrightarrow$ (3.) is Proposition 4.2. (3.) $\Rightarrow$ (4.) is Proposition 3.2. (4.) $\Rightarrow$ (5.) is Proposition 5.6. (5.) $\Rightarrow$ (6.) is trivial. (6.) $\Rightarrow$ (1.) is Theorem 6.10. ■

Corollary 7.2 *The sequent calculus* **S4F**$_G$ *admits cut-elimination.*

Corollary 7.3 *The logic* **S4F** *is decidable.*

8 Adding Continuity: S4C

In our definition of a topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for the language $\mathcal{L}_{\Box a}$, we place no restrictions on the function $f : X \to X$, other than totality. The language itself is rich enough to express various properties of f, notably the continuity of f with respect to the topology $\mathcal{T}$. We call the scheme

$$\textbf{Cont}: \quad [a]\Box\varphi \to \Box[a]\varphi$$

the *continuity axiom*, in virtue of the following proposition.

Proposition 8.1 [Kur66] I,§13; [RS63] III,§3.
Let $\mathfrak{T} = (X, \mathcal{T}, f)$ be a topological structure for $\mathcal{L}_{\Box a}$. Then the following are equivalent:

(a) *for each $\varphi \in \mathcal{L}_{\Box a}$, $\mathfrak{T} \models [a]\Box\varphi \to \Box[a]\varphi$;*

(b) *for each $\varphi \in \mathcal{L}_{\Box a}$, $\mathfrak{T} \models [a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi$;*

(c) *the function $f : X \to X$ is continuous with respect to the topology $\mathcal{T}$.*

Proof. Let φ be any formula of $\mathcal{L}_{\Box a}$, let ξ be any valuation for $\mathfrak{T}$, and let $A = \|\varphi\|_\xi \subseteq X$. Then

$$\|[a]\Box\varphi \to \Box[a]\varphi\|_\xi = X \;\text{ iff }\; f^{-1}(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

and

$$\|[a]\Box\varphi \leftrightarrow \Box[a]\Box\varphi\|_\xi = X \;\text{ iff }\; f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A)))$$

Now the following equivalence is immediate:

$$\begin{array}{lll} & \text{(b)}: & f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \;\text{ for all } A \subseteq X \\ \text{iff} & \text{(c)}: & f^{-1}(U) = int_{\mathcal{T}}(f^{-1}(U)) \;\text{ for all } U \in \mathcal{T} \\ & & \text{i.e. } f \text{ is continuous w.r.t. the topology } \mathcal{T} \end{array}$$

since $U \in \mathcal{T}$ iff $U = int_{\mathcal{T}}(U)$, and for any $A \subseteq X$, we have $int_{\mathcal{T}}(A) = U$ for some $U \in \mathcal{T}$. So rewriting

$$\text{(a)}: \quad f^{-1}(int_{\mathcal{T}}(A)) \subseteq int_{\mathcal{T}}(f^{-1}(A)) \;\text{ for all } A \subseteq X$$

it suffices to show that (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a).

Assume (a) holds. Then for any $U \in \mathcal{T}$, we have $U = int_{\mathcal{T}}(U)$, hence

$$int_{\mathcal{T}}(f^{-1}(U)) \subseteq f^{-1}(U) = f^{-1}(int_{\mathcal{T}}(U)) \subseteq int_{\mathcal{T}}(f^{-1}(U))$$

and thus

$$f^{-1}(U) = int_{\mathcal{T}}(f^{-1}(U))$$

so (a) $\Rightarrow$ (c).

Now, for any $A \subseteq X$, we have $int_{\mathcal{T}}(A) \subseteq A$, hence applying $int_{\mathcal{T}} \circ f^{-1}$, we have

$$int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

Thus if (b) holds, we have

$$f^{-1}(int_{\mathcal{T}}(A)) = int_{\mathcal{T}}(f^{-1}(int_{\mathcal{T}}(A))) \subseteq int_{\mathcal{T}}(f^{-1}(A))$$

hence (b) $\Rightarrow$ (a), as required. ■

The preceding proposition gives us an alternative, equivalent version of the continuity axiom, namely:

$$\mathbf{Cont}^* : \quad [a]\Box\varphi \to \Box[a]\Box\varphi$$

It is also readily established that over the Hilbert system $\mathbf{S4F}_H$, the schemes **Cont** and **Cont*** are provably equivalent. The **Cont*** scheme will be appealed to in devising a sequent calculus rule capturing continuity.

From [RS63] and [Kur66], the converse of the **Cont** scheme,

$$\mathbf{Open} : \quad \Box[a]\varphi \to [a]\Box\varphi$$

characterizes the *open mapping* property. All instances of the **Open** scheme are true in a topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$, exactly when the function $f : X \to X$ is such that for all $U \in \mathcal{T}$, the image $f(U) \in \mathcal{T}$, since the latter condition holds exactly when

$$int_{\mathcal{T}}(f^{-1}(A)) \subseteq f^{-1}(int_{\mathcal{T}}(A)) \text{ for all } A \subseteq X;$$

see [RS63], III,§3, p. 99, and [Kur66], I,§13,XIV. Thus the conjunction of the schemes **Cont** and **Open**, namely:

$$\Box[a]\varphi \leftrightarrow [a]\Box\varphi$$

characterizes *continuous and open* maps $f : X \to X$; equivalently, the set map $f^{-1} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a *(topological) homomorphism* of the topological Boolean algebra $\mathfrak{B}_{\mathcal{T}}(X) = (\mathcal{P}(X), \cup, \cap, -, X, \emptyset, int_{\mathcal{T}})$ into itself ([RS63], III,§3).

In this study, our chief interest is in continuity. Next, we characterize the Kripke models which satisfy the continuity axiom.

Proposition 8.2 *Let $\mathcal{K} = (W, R, F)$ be a Kripke frame for $\mathcal{L}_{\Box a}$. Then the following are equivalent:*

(a) *for each $\varphi \in \mathcal{L}_{\Box a}$, $\mathcal{K} \Vdash [a]\Box\varphi \to \Box[a]\varphi$;*

(b) *the function $F : W \to W$ satisfies the condition:*

$$(w, v) \in R \;\text{ implies }\; (F(w), F(v)) \in R$$

for all $w, v \in W$.

Proof. For (b) $\Rightarrow$ (a), fix $\varphi \in \mathcal{L}_{\Box a}$, $w \in W$ and a valuation η for $\mathcal{K}$. Then

$$\begin{array}{ll} & w \nVdash_\eta [a]\Box\varphi \to \Box[a]\varphi \\ \Leftrightarrow & w \Vdash_\eta [a]\Box\varphi \text{ and } w \nVdash_\eta \Box[a]\varphi \\ \Leftrightarrow & \text{for all } x \in W, \text{ if } (F(w), x) \in R \text{ then } x \Vdash_\eta \varphi, \\ & \text{and for some } v \in W, \ (w, v) \in R \text{ and } F(v) \nVdash_\eta \varphi \\ \Rightarrow & \text{for some } v \in W, \ (w, v) \in R, \\ & \text{but for all } x \in W, \text{ if } (F(w), x) \in R \text{ then } x \neq F(v) \\ \Leftrightarrow & \text{for some } v \in W, \ (w, v) \in R \text{ but } (F(w), F(v)) \notin R \end{array}$$

For (a) $\Rightarrow$ (b), suppose (b) is false, so there exists $w, v, u, z \in W$ such that $(w, v) \in R$, $u = F(w)$, $z = F(v)$ and $(u, z) \notin R$. (By reflexivity, $u \neq z$, so W must have at least 2 elements, and so be non-degenerate.) Choose any $p \in AP$ and define $\eta : W \to \mathcal{P}(AP)$ by

$$\eta(x) = \begin{cases} \{p\} & \text{if } (u, x) \in R \\ \emptyset & \text{otherwise} \end{cases}$$

By construction of η, $(u, z) \notin R$ implies $z \nVdash_\eta p$, hence $F(v) \nVdash_\eta p$, since $z = F(v)$, and so $v \nVdash_\eta [a]p$. Since $(w, v) \in R$, this means $w \nVdash_\eta \Box[a]p$.
Our chosen valuation η also gives us $x \Vdash_\eta p$ for all $x \in W$ such that $(u, x) \in R$, hence $u \Vdash_\eta \Box p$; since $u = F(w)$, we have $w \Vdash_\eta [a]\Box p$.
Hence $w \nVdash_\eta [a]\Box p \to \Box[a]p$. ■

For comparative purposes, note that a Kripke frame $\mathcal{K} = (W, R, F)$ forces all instances of the **Open** scheme exactly when the condition:

$$(F(w), u) \in R \;\Rightarrow\; (\exists v \in W)[\, F(v) = u \text{ and } (w, v) \in R\,] \quad (F\text{−open})$$

holds for all $w, u \in W$. This condition is properly stronger than the converse of R-monotonicity:

$$(F(w), F(v)) \in R \;\Rightarrow\; (w, v) \in R$$

since the (F−open) condition can fail when F is not surjective; i.e. there is a $u \in W$ such that $u \neq F(v)$ for all $v \in W$.

Definition 8.3 *A topological structure $\mathfrak{T} = (X, \mathcal{T}, f)$ for $\mathcal{L}_{\Box a}$ is called* continuous *iff f is continuous with respect to $\mathcal{T}$.*

A Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$ is called continuous *iff F satisfies the condition:*

$$(w, v) \in R \;\text{ implies }\; (F(w), F(v)) \in R$$

for all $w, v \in W$; i.e. F is R-monotone.

Proposition 8.4 *For all formulas φ of $\mathcal{L}_{\Box a}$,*
if $\mathfrak{T} \models \varphi$ for all continuous topological structures $\mathfrak{T}$ for $\mathcal{L}_{\Box a}$,
then $\mathcal{K} \Vdash \varphi$ for all continuous Kripke frames $\mathcal{K}$ for $\mathcal{L}_{\Box a}$.

Proof. From Proposition 5.6, it suffices to show that for each continuous Kripke frame $\mathcal{K} = (W, R, F)$ for $\mathcal{L}_{\Box a}$, the induced topological structure $\mathfrak{T}_{\mathcal{K}} = (W, \mathcal{T}_R, F)$ is such that F is continuous w.r.t. the topology $\mathcal{T}_R$. Now for arbitrary $A \subseteq W$ and $w \in W$, we have:

$$\begin{array}{ll} & w \in F^{-1}(int_{\mathcal{T}_R}(A)) \\ \Leftrightarrow & F(w) \in int_{\mathcal{T}_R}(A) \\ \Leftrightarrow & (\forall z \in W)[\ (F(w), z) \in R \Rightarrow z \in A\ ] \\ \Rightarrow & (\forall v \in W)[\ (w, v) \in R \Rightarrow F(v) \in A\ ] \qquad (*) \\ \Leftrightarrow & (\forall v \in W)[\ (w, v) \in R \Rightarrow v \in F^{-1}(A)\ ] \\ \Leftrightarrow & w \in int_{\mathcal{T}_R}(F^{-1}(A)) \end{array}$$

with the implication $(*)$ a consequence of: $(w, v) \in R \Rightarrow (F(w), F(v)) \in R$. (It is also readily verified that the converse also holds: F is continuous with respect to $\mathcal{T}_R$ implies F is R-monotone.) ■

Definition 8.5 *The Hilbert-style proof system for the logic* **S4C** *has as its axiom schemes those of* **S4F** *(Definition 3.1) together with all instances of the scheme*

$$\textbf{Cont}: \quad [a]\Box\varphi \to \Box[a]\varphi$$

in the language $\mathcal{L}_{\Box a}$; the inference rules are the same as those of **S4F**.
We write

$$\textbf{S4C} \vdash_H \varphi$$

or say φ is $\textbf{S4C}_H$ provable, if the formula $\varphi \in \mathcal{L}_{\Box a}$ has an **S4C** *Hilbert-style derivation.*

The following are derivable in $\textbf{S4C}_H$, for any formula $\varphi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$.

$$\begin{array}{ll} [a]^k\textbf{Cont}: & [a]^k\Box\varphi \to \Box[a]^k\varphi \\ [a]^k\Diamond\textbf{Cont}: & \Diamond[a]^k\varphi \to [a]^k\Diamond\varphi \end{array}$$

The following is an admissible inference rule in $\textbf{S4C}_H$, for any formulas $\varphi, \psi, \chi \in \mathcal{L}_{\Box a}$ and $k, l \in \mathbb{N}$:

Continuous
Hoare composition: $$\frac{\varphi \to [a]^k\Box\chi,\ \chi \to [a]^l\Box\psi}{\varphi \to [a]^{k+l}\Box\psi}$$

Proposition 8.6 Soundness of **S4C** Hilbert-style proof system
For all formulas φ *of* $\mathcal{L}_{\Box a}$*, if* $\mathbf{S4C} \vdash_H \varphi$*,*
then $\mathfrak{T} \models \varphi$ *for all continuous topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$
and $\mathcal{K} \Vdash \varphi$ *for all continuous Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$*.*

Proof. Immediate from Propositions 3.2, 8.1 and 8.2. ■

Definition 8.7 *The Gentzen-style sequent calculus for the logic* **S4C** *has the same axioms and rules as those for* **S4F** *(Definition 4.1), and in addition, the rule:*

$$(\mathbf{Cont_G}): \quad \frac{\Box[a]\Box\varphi, \Gamma \Rightarrow \Delta}{[a]\Box\varphi, \Gamma \Rightarrow \Delta}$$

The first point of note is that this new rule violates the sub-formula property, but it does so in a manageable way. To compensate, we have to deal with a larger class of psuedo-sub-formulas of a sequent.

Definition 8.8 *For each sequent* $\Gamma_0 \Rightarrow \Delta_0$ *of* $\mathcal{L}_{\Box a}$*, define*

$$\begin{aligned} & \Box\!-\!SubForm(\Gamma_0 \cup \Delta_0) \\ \mathrel{\mathring{=}} \; & SubForm(\Gamma_0 \cup \Delta_0) \cup \{\{\Box\varphi \mid \varphi \in SubForm(\Gamma_0 \cup \Delta_0)\}\} \end{aligned}$$

where $SubForm(\Gamma_0 \cup \Delta_0)$ *and* $\Box$*-*$SubForm(\Gamma_0 \cup \Delta_0)$ *are multisets of formulas*[4].

Proposition 8.9 Equivalence of Sequent Calculus and Hilbert-style proof system for **S4C**
Let Γ *and* Δ *be multisets of formulas of* $\mathcal{L}_{\Box a}$*, and let* φ *be any formula of* $\mathcal{L}_{\Box a}$*.*

(i) *If* $\mathbf{S4C} \vdash_G \Gamma \Rightarrow \Delta$ *then* $\mathbf{S4C} \vdash_H \bigwedge \Gamma \to \bigvee \Delta$*.*

(ii) *If* $\mathbf{S4C} \vdash_H \varphi$ *then* $\mathbf{S4C} \vdash_G \Rightarrow \varphi$*.*

Proof. For (i), beyond the proof of part (i) of Proposition 4.2, we need only consider the case where the last rule applied in the $\mathbf{S4C}_G$ derivation of $\Gamma \Rightarrow \Delta$ is the new $(\mathbf{Cont_G})$ rule. So assume Γ is $[a]\Box\varphi, \Gamma'$ and the sequent $\Gamma \Rightarrow \Delta$ is derived from $\Box[a]\Box\varphi, \Gamma' \Rightarrow \Delta$ by the $(\mathbf{Cont_G})$ rule. By the induction hypothesis, $\mathbf{S4C} \vdash_H \Box[a]\Box\varphi \wedge (\bigwedge \Gamma') \to \bigvee \Delta$. Then

[4] As in the discussion following Definition 6.1

1.	$\Box[a]\Box\varphi \wedge (\bigwedge \Gamma') \to \bigvee \Delta$	induction hypothesis
2.	$\Box\varphi \to \Box\Box\varphi$	axiom $\Box$**4**
3.	$[a](\Box\varphi \to \Box\Box\varphi)$	from 2. by $[a]-$necessitation
4.	$[a]\Box\varphi \to [a]\Box\Box\varphi$	from 3. by $[a]$**K**
5.	$[a]\Box\Box\varphi \to \Box[a]\Box\varphi$	axiom **Cont**
6.	$[a]\Box\varphi \to \Box[a]\Box\varphi$	from 4. and 5.
7..	$[a]\Box\varphi \wedge (\bigwedge \Gamma') \to \bigvee \Delta$	from 1. and 6. by propositional logic

For (ii), beyond the proof of part (ii) of Proposition 4.2, we only need show that the **Cont** axiom $[a]\Box\varphi \to \Box[a]\varphi$ is derivable in $\mathbf{S4C}_G$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\varphi \Rightarrow \varphi \quad (Axiom)}{\Box\varphi \Rightarrow \varphi} \ (\Box \Rightarrow)}{[c]\Box\varphi \Rightarrow [c]\varphi} \ ([a] \Rightarrow [a])}{\Box[c]\Box\varphi \Rightarrow [c]\varphi} \ (\Box \Rightarrow)}{\Box[c]\Box\varphi \Rightarrow \Box[c]\varphi} \ (\Rightarrow \Box)}{[c]\Box\varphi \Rightarrow \Box[c]\varphi} \ (\mathbf{Cont}_G)}{\Rightarrow [a]\Box\varphi \to \Box[a]\varphi} \ (\Rightarrow\to)$$

■

Observe that the scheme

$$[a]^k\mathbf{Cont}^* : \quad [a]^k[a]\Box\varphi \to [a]^k\Box[a]\Box\varphi$$

is derivable in $\mathbf{S4C}_H$: the derivation can be extracted from the proof of part (i) of the previous proposition, together with k applications of $[a]$-necessitation and an appeal to $[a]^k\mathbf{K}\leftrightarrow$.

Proposition 8.10 *Let Γ and Δ be multisets of formulas of $\mathcal{L}_{\Box a}$, let φ be a formula of $\mathcal{L}_{\Box a}$, and let $k \in N$. The following rule is admissible in the (cut-free) sequent calculus $\mathbf{S4C}_{G^-}$.*

$$([a]^k\mathbf{Cont_G}) : \quad \frac{[a]^k\Box[a]\Box\varphi, \Gamma \Rightarrow \Delta}{[a]^k[a]\Box\varphi, \Gamma \Rightarrow \Delta}$$

Proof. Again, as in 4.3, a straightforward strategy should be to first apply the rule $\mathbf{Cont}_G$ and then deal with the $[a]^k$ prefix. Without any loss of generality we may consider the case $k = 1$. Let $[a]\Box[a]\Box\varphi, \Gamma \Rightarrow \Delta$ be derived in $\mathbf{S4C}_{G}$- and let $\mathcal{D}$ be a corresponding derivation. Consider a node in $\mathcal{D}$ where the formula $[a]\Box[a]\Box\varphi$ was introduced first. There are three possibilities for a sequent assigned to this node: it is an axiom, an instant of the weakening or the $[a] \Rightarrow [a]$ rule. Let us treat the latter. The node under consideration is

$$\frac{\Box[a]\Box\varphi, \Gamma' \Rightarrow \Delta'}{[a]\Box[a]\Box\varphi, [a]\Gamma' \Rightarrow [a]\Delta'}$$

We replace this node by a pair of nodes

$$\dfrac{\dfrac{\Box[a]\Box\varphi, \Gamma' \Rightarrow \Delta'}{[a]\Box\varphi, \Gamma' \Rightarrow \Delta'}\ (\mathbf{Cont}_G)}{[a][a]\Box\varphi, [a]\Gamma' \Rightarrow [a]\Delta'}\ ([a] \Rightarrow [a])$$

Now replace everywhere in the path from this node to the root sequent all corresponding occurrences of $[a]\Box[a]\Box\varphi$ by $[a][a]\Box\varphi$. Perform this operation with all the nodes where $[a]\Box[a]\Box\varphi$ was introduced, adjust some weakenings and get an $\mathbf{S4C}_{G^-}$ derivation of the desired sequent $[a][a]\Box\varphi, \Gamma \Rightarrow \Delta$.

We leave the remaining cases to a reader as routine exercises. ■

9 Kripke Completeness for S4C

To prove completeness for **S4C**, we modify the proof of Kripke completeness (and the finite model property) for **S4F** by further strengthening the notion of saturation to behave well with new $\mathbf{Cont_G}$ rule, and force the "$[a]$-stripping" *Strip* function to be monotone with respect to the accessibility relation:

$$\Big((\Gamma \Rightarrow \Delta), (\Gamma' \Rightarrow \Delta')\Big) \in R \quad \text{iff} \quad [\ \Box\varphi \in \Gamma \ \text{ implies } \Box\varphi \in \Gamma'\ ]$$

Let's start with a stronger notion of saturation.

Definition 9.1 *A sequent $\Gamma \Rightarrow \Delta$ in the language $\mathcal{L}_{\Box a}$ is called* **S4C** saturated *iff each the following conditions hold:*

(1.) *if $[a]^k(\varphi \to \psi) \in \Gamma$ then either $[a]^k\psi \in \Gamma$ or $[a]^k\varphi \in \Delta$;*

(2.) *if $[a]^k(\varphi \to \psi) \in \Delta$ then both $[a]^k\varphi \in \Gamma$ and $[a]^k\psi \in \Delta$;*

(3.) *if $[a]^k\Box\varphi \in \Gamma$ then $[a]^k\varphi \in \Gamma$;*

(4.) *if $[a]^k[a]\Box\varphi \in \Gamma$ then $[a]^k\Box[a]\Box\varphi \in \Gamma$,*

for all $\varphi, \psi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$.

Note that if $\Gamma \Rightarrow \Delta$ is **S4C** saturated, then $\Gamma \Rightarrow \Delta$ is **S4F** saturated, since **S4F** saturation is just clauses (1.), (2.) and (3.) of **S4C** saturation. Clause (4.) of **S4C** saturation is reflected in the $\mathbf{S4F}_{G^-}$ admissible rule given in Proposition 8.10.

Lemma 9.2 S4C Saturation

For each sequent $\Gamma_0 \Rightarrow \Delta_0$ *in the language* $\mathcal{L}_{\Box a}$,
if $\mathbf{S4C} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$*, then there is an* **S4C** *saturated sequent* $\Gamma \Rightarrow \Delta$ *such that*

(a) $\Gamma_0 \subseteq \Gamma \subseteq \Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$;

(b) $\Delta_0 \subseteq \Delta \subseteq \Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$;

(c) $\mathbf{S4C} \nvdash_{G-} \Gamma \Rightarrow \Delta$.

Moreover, by determinizing the algorithm which produces such a saturated sequent from input $\Gamma_0 \Rightarrow \Delta_0$*, we may take the output* $\Gamma \Rightarrow \Delta$ *to be unique, and denote it* $Sat_{\mathbf{S4C}}(\Gamma_0 \Rightarrow \Delta_0)$*,* the **S4C** *saturation of* $\Gamma_0 \Rightarrow \Delta_0$.

Proof. The saturation algorithm and its verification are analogous with those in the proof of Lemma 6.2.

In the **Marking(**$\Gamma \Rightarrow \Delta$**)** sub-routine, add an extra line**:**

- For each occurrence of a formula $[a]^k[a]\Box\varphi$ in Γ, if there is no occurrence of $[a]^k\Box[a]\Box\varphi$ in Γ then mark the $[a]^k[a]\Box\varphi$ with "0"; otherwise, mark it with "1".

In the main body of the algorithm, we add extra clauses:

4. Antecedent $[a]^k[a]\Box$: If Γ contains an occurrence of a formula $[a]^k[a]\Box\varphi$ marked "0", put a check mark "✓" next to the current node, then create one child node:

$$\begin{array}{c} \Gamma, [a]^k\Box[a]\Box\varphi \Rightarrow \Delta \\ | \\ \Gamma \Rightarrow \Delta \quad \checkmark \end{array}$$

labelled $\Gamma, [a]^k\Box[a]\Box\varphi \Rightarrow \Delta$. Run the sub-routine **Marking(**$\Gamma, [a]^k\Box[a]\Box\varphi \Rightarrow \Delta$**)**. Select the child node labelled $\Gamma, [a]^k\Box[a]\Box\varphi \Rightarrow \Delta$ as the new *current node.*

If Γ contains no occurrences of any formula $[a]^k[a]\Box\varphi$ marked "0", then proceed to **5.**

5. Terminate and return the label of the current node, $\Gamma \Rightarrow \Delta$ (which does NOT have a check mark "✓") as *the* saturation of $\Gamma_0 \Rightarrow \Delta_0$, i.e. $Sat_{\mathbf{S4C}}(\Gamma_0 \Rightarrow \Delta_0) = \Gamma \Rightarrow \Delta$

As before, the saturation algorithm must terminate because $\Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$ is finite and there is at most two branches at each step.

It is immediate from the construction that if $\Gamma \Rightarrow \Delta = Sat_{\mathbf{S4C}}(\Gamma_0 \Rightarrow \Delta_0)$ then

(a) $\Gamma_0 \subseteq \Gamma \subseteq \Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$ and

(b) $\Delta_0 \subseteq \Delta \subseteq \Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$
hold, and

(c) $\mathbf{S4C} \nvdash_{G-} \Gamma \Rightarrow \Delta$

by the same argument as in Lemma 6.2. ■

Next, we summarize the relevant properties of the *Strip* function in this setting.

Lemma 9.3 *Let $\Gamma \Rightarrow \Delta$ be a sequent of $\mathcal{L}_{\Box a}$, and let $(\Gamma' \Rightarrow \Delta') = Strip(\Gamma \Rightarrow \Delta)$. Then for all $\varphi \in \mathcal{L}_{\Box a}$ and $k \in \mathbb{N}$,*

(i) $\Gamma' \subseteq SubForm(\Gamma)$ *and* $\Delta' \subseteq SubForm(\Delta)$;

(ii) $[a]^{k+1}\varphi \in \Gamma$ *iff* $[a]^k\varphi \in \Gamma'$;

(iii) $[a]^{k+1}\varphi \in \Delta$ *iff* $[a]^k\varphi \in \Delta'$;

(iv) *if* $\mathbf{S4C} \nvdash_{G-} \Gamma \Rightarrow \Delta$, *then* $\mathbf{S4C} \nvdash_{G-} \Gamma' \Rightarrow \Delta'$;

(v) if $\Gamma \Rightarrow \Delta$ is **S4C** *saturated, then $\Gamma' \Rightarrow \Delta'$ is also* **S4C** *saturated.*

Proof. Properties (i), (ii) and (iii) are as in Lemma 6.4, and the argument for (iv) is identical to that in the proof of that lemma. For (v), we only need check clause (4.) of **S4C** saturation. Suppose $\Gamma \Rightarrow \Delta$ is **S4C** saturated, and consider $\Gamma' \Rightarrow \Delta'$. Then

$$\begin{array}{lll} & [a]^k[a]\Box\varphi \in \Gamma' & \\ \Leftrightarrow & [a]^{k+1}[a]\Box\varphi \in \Gamma & \text{by (ii)} \\ \Rightarrow & [a]^{k+1}\Box[a]\Box\varphi \in \Gamma & \text{by } \mathbf{S4C} \text{ saturation of } \Gamma \Rightarrow \Delta \\ \Leftrightarrow & [a]^k\Box[a]\Box\varphi \in \Gamma' & \text{by (ii)} \end{array}$$

Hence $\Gamma' \Rightarrow \Delta'$ is **S4C** saturated. ■

Definition 9.4 *For each sequent $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$, define Sub-***S4C***$(\Gamma_0 \Rightarrow \Delta_0)$ to be the set of all sequents $\Gamma \Rightarrow \Delta$ in $\mathcal{L}_{\Box a}$ satisfying the three properties:*

- $\Gamma \Rightarrow \Delta$ *is* **S4C** *saturated;*
- $\Gamma \cup \Delta \subseteq \Box\text{-}SubForm(\Gamma_0 \cup \Delta_0)$;
- $\mathbf{S4C} \nvdash_{G-} \Gamma \Rightarrow \Delta$.

Definition 9.5 *For each sequent $\Gamma_0 \Rightarrow \Delta_0$ in the language $\mathcal{L}_{\Box a}$, we define a Kripke frame $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)} = (W, R, F)$ for $\Gamma_0 \Rightarrow \Delta_0$ as follows:*

- $W = Sub\text{-}\mathbf{S4C}(\Gamma_0 \Rightarrow \Delta_0)$;
- $\big((\Gamma \Rightarrow \Delta), (\Gamma' \Rightarrow \Delta')\big) \in R$ *iff* $\Box\varphi \in \Gamma$ *implies* $\Box\varphi \in \Gamma'$
- $F = Strip$

The Kripke frame $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ *is called the* **S4C** saturation frame *for* $\Gamma_0 \Rightarrow \Delta_0$.
Define the canonical valuation $\eta : W \to \mathcal{P}(\mathbf{PV})$ *for* $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ *by*

$$p \in \eta(\Gamma \Rightarrow \Delta) \ \text{ iff } \ p \in \Gamma$$

By Lemma 9.3, $W = Sub\text{-}\mathbf{S4C}(\Gamma_0 \Rightarrow \Delta_0)$ is closed under $F = Strip$, and R is reflexive and transitive, hence $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ is a Kripke frame for $\mathcal{L}_{\Box a}$. Our real interest is in the monotonicity of $Strip$ with respect to R.

Lemma 9.6 *Let* $\Gamma_0 \Rightarrow \Delta_0$ *be a sequent of* $\mathcal{L}_{\Box a}$ *such that* **S4C** $\nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$, *and let* $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ *be the* **S4C** *saturation frame for* $\Gamma_0 \Rightarrow \Delta_0$, *as in Definition 9.5.*
Then $F = Strip$ *is monotone with respect to the relation* R, *where*

$$((\Gamma_1 \Rightarrow \Delta_2), (\Gamma_2 \Rightarrow \Delta_2)) \in R \quad \Leftrightarrow \quad \Box\varphi \in \Gamma_1 \text{ implies } \Box\varphi \in \Gamma_2$$

Hence $\mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)}$ *is a continuous Kripke frame.*

Proof. Assume $(\Gamma_1 \Rightarrow \Delta_1), (\Gamma_2 \Rightarrow \Delta_2) \in W^C_{(\Gamma_0 \Rightarrow \Delta_0)}$, and $((\Gamma_1 \Rightarrow \Delta_1), (\Gamma_2 \Rightarrow \Delta_2)) \in R$. Let $Strip(\Gamma_i \Rightarrow \Delta_i) = (\Gamma_i' \Rightarrow \Delta_i')$, for $i = 1, 2$. Then fix $\varphi \in \mathcal{L}_{\Box a}$. Then

$$\begin{array}{lll}
 & \Box\varphi \in \Gamma_1' & \\
\Leftrightarrow & [a]\Box\varphi \in \Gamma_1 & \text{by (ii) of Lemma 9.3} \\
\Rightarrow & \Box[a]\Box\varphi \in \Gamma_1 & \text{by } \mathbf{S4C} \text{ saturation, clause (4.) with } k = 0 \\
\Rightarrow & \Box[a]\Box\varphi \in \Gamma_2 & \text{by definition of } R \\
\Rightarrow & [a]\Box\varphi \in \Gamma_2 & \text{by } \mathbf{S4C} \text{ saturation, clause (3.) with } k = 0 \\
\Leftrightarrow & \Box\varphi \in \Gamma_2' & \text{by (ii) of Lemma 9.3}
\end{array}$$

Hence $((\Gamma_1' \Rightarrow \Delta_1'), (\Gamma_2' \Rightarrow \Delta_2'))$, as required. ■

Lemma 9.7 Main Semantic Lemma for **S4C**
Let $\Gamma_0 \Rightarrow \Delta_0$ *be any sequent in* $\mathcal{L}_{\Box a}$, *let* $\mathcal{K} = \mathcal{K}_{(\Gamma_0 \Rightarrow \Delta_0)} = (W, R, F)$ *be the* **S4C** *saturation frame for* $\Gamma_0 \Rightarrow \Delta_0$, *and let* η *be the canonical valuation for* $\mathcal{K}$ *as in Definition 9.5.*
Then for all $(\Gamma \Rightarrow \Delta) \in W$ *and for all formulas* φ *in* $\mathcal{L}_{\Box a}$, *we have:*

$$\begin{array}{lll}
\varphi \in \Gamma & \text{implies} & (\Gamma \Rightarrow \Delta) \Vdash_\eta \varphi \\
\varphi \in \Delta & \text{implies} & (\Gamma \Rightarrow \Delta) \nVdash_\eta \varphi
\end{array}$$

Proof. Duplicate the proof of Lemma 6.9, replacing **S4F** with **S4C** and $SubForm$ with $\Box$-$SubForm$ in the analysis of $\Box$ in the succedent. ■

Theorem 9.8 Kripke completeness and finite model property for **S4C**
Let $\Gamma_0 \Rightarrow \Delta_0$ *be any sequent in* $\mathcal{L}_{\Box a}$.
If $\mathbf{S4C} \nvdash_{G-} \Gamma_0 \Rightarrow \Delta_0$,
then there is a finite continuous Kripke frame $\mathcal{K}$ *and valuation* η *for* $\mathcal{K}$ *such that* $(\mathcal{K}, \eta) \nVdash \bigwedge \Gamma_0 \to \bigvee \Delta_0$.

Proof. Same as the proof of Theorem 6.10. ■

10 Consolidation Theorems for S4C

As for **S4F**, we consolidate the major results of previous sections.

Theorem 10.1 *For all multisets* Γ, Δ *of formulas of* $\mathcal{L}_{\Box a}$*, the following are equivalent:*

(1.) $\mathbf{S4C} \vdash_{G-} \Gamma \Rightarrow \Delta$

(2.) $\mathbf{S4C} \vdash_{G} \Gamma \Rightarrow \Delta$

(3.) $\mathbf{S4C} \vdash_{H} \bigwedge \Gamma \to \bigvee \Delta$

(4.) $\mathfrak{T} \models \bigwedge \Gamma \to \bigvee \Delta$ *for all continuous topological structures* $\mathfrak{T}$ *for* $\mathcal{L}_{\Box a}$,

(5.) $\mathcal{K} \Vdash \bigwedge \Gamma \to \bigvee \Delta$ *for all continuous Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$,

(6.) $\mathcal{K} \Vdash \bigwedge \Gamma \to \bigvee \Delta$ *for all finite continuous Kripke frames* $\mathcal{K}$ *for* $\mathcal{L}_{\Box a}$.

Proof. (1.) $\Rightarrow$ (2.) is trivial. (2.) $\Leftrightarrow$ (3.) is Proposition 8.9. (3.) $\Rightarrow$ (4.) is Proposition 8.6. (4.) $\Rightarrow$ (5.) is Proposition 8.4. (5.) $\Rightarrow$ (6.) is trivial. (6.) $\Rightarrow$ (1.) is Theorem 9.8. ■

Corollary 10.2 *The sequent calculus* $\mathbf{S4C}_G$ *admits cut-elimination.*

Corollary 10.3 *The logic* **S4C** *is decidable.*

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