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Topological Models for Justification Logic

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Gödel's modal logic of provability.

Gödel (1933) introduced the modal logic S4 as the system axiomatizing provability in classical mathematics:

Axioms and rules of classical propositional logic

$\Box(F\rightarrow G)\rightarrow(\Box F\rightarrow\Box G)$ *Normality*

$\Box F\rightarrow F$ *Reflexivity*

$\Box F\rightarrow\Box\Box F$ *Transitivity*

Necessitation Rule: $\dfrac{\vdash F}{\vdash \Box F}$

Gödel's provability semantics for modality

Gödel also considered the interpretation of $\Box F$ as

F *is provable in Peano Arithmetic* PA

and noticed that this semantics is inconsistent with S4.

Indeed, $\Box(\Box F \rightarrow F)$ can be derived in S4. On the other hand, interpreting $\Box$ as the predicate *Provable* of formal provability in Peano Arithmetic PA and F as *falsum* $\perp$, converts this formula into the false statement that the consistency of PA is internally provable in PA:

Provable (*Consis PA*) .

Gödel's paper left open two questions

1. What is the precise provability semantics for S4?

‘**a provability calculus without a semantics**’

Was answered by the first author (1995) whose Logic of Proofs LP provided a semantics of explicit proofs for S4.

2. What is the logic of formal provability *Provable*?

‘**a provability semantics without a calculus**’

Was answered by Solovay (1976), who found the logic of formal provability GL.

The provability logic GL

Axioms and rules of classical propositional logic

$\Box(F \to G) \to (\Box F \to \Box G)$ *Normality*

$\Box(\Box F \to F) \to \Box F$ *Löb Axiom*

$\Box F \to \Box\Box F$ *Transitivity*

Necessitation Rule: $\dfrac{\vdash F}{\vdash \Box F}$

Semantics of formal provability

Formal provability **interpretation** of a modal language is a mapping $*$ from the set of modal formulas to the set of arithmetical sentences such that $*$ agrees with Boolean connectives and constants and

$$(\Box G)^* = \textit{Provable } G^* \ .$$

Solovay's completeness theorem:

$\mathsf{GL} \vdash F$ iff *for all interpretations* $*$, $\mathsf{PA} \vdash F^*$.

Alternative Gödel's format for provability

In his lecture in Vienna in 1938 Gödel mentioned a possibility of building an explicit version of S4 with basic propositions "t is a proof of F":

$$\mathit{Proof}(t, F)$$

This Gödel's lecture remained unpublished until 1995. By that time the full Logic of Proofs was already discovered by the first author.

The principal observation of the Logic of Proofs

Explicit reflection $\mathit{Proof}(p, F) \rightarrow F$ for each specific derivation p is provable in PA. Indeed,

• If $\mathit{Proof}(p, F)$ holds, then F is evidently provable in PA, and so is the formula $\mathit{Proof}(p, F) \rightarrow F$.

• If $\neg \mathit{Proof}(p, F)$ holds, then it is provable in PA (since $\mathit{Proof}(x, y)$ is decidable) and $\mathit{Proof}(p, F) \rightarrow F$ is again provable.

Logic of Proofs LP: the language

Proof polynomials are terms built from *proof variables* $x, y, z, \ldots$ and *proof constants* $a, b, c, \ldots$ by means of two binary operations: *application* '$\cdot$' and *choice* '$+$', and one unary *proof checker* '!'.

Using t to stand for any proof polynomial and S for any sentence letter, the **formulas of the Logic of Proofs** are defined by the grammar

$$A = S \mid A \to A \mid A \wedge A \mid A \vee A \mid \neg A \mid t{:}A \ .$$

Logic of Proofs LP

The standard axioms and rules of classical propositional logic

$t{:}(F \to G) \;\to\; (s{:}F \to (t \cdot s){:}G)$ *Application*

$t{:}F \;\to\; !t{:}(t{:}F)$ *Proof Checker*

$s{:}F \to (s+t){:}F, \quad t{:}F \to (s+t){:}F$ *Choice*

$t{:}F \to F$ *Reflexivity*

$\vdash c{:}A$, where A is an axiom and c is a proof constant

– *Constant Specification Rule*

Realization of S4 **in the Logic of Proofs** LP

S4 *is the forgetful projection of* LP, *i.e.,*

1. *The forgetful projection of* LP *is* S4*-compliant.*

2. *For each theorem F of* S4 *one can recover a witness (proof polynomial) for each occurrence of $\Box$ in F in such a way that the resulting formula F^r is derivable in* LP.

Realization gives a semantics of proofs for S4**.**

$$\mathsf{S4} \vdash F \quad \Leftrightarrow \quad \exists r \ \mathsf{LP} \vdash F^r$$

Provability semantics of LP

Interpretations respect Boolean connectives and

$$(p{:}F)^* \quad = \quad \mathit{Proof}(p^*, F^*).$$

Completeness theorem (by S.A.):

LP *derives all valid logical principles in its language*

Joining languages GL and LP

Artemov (1994), Yavorskaya (1997), Nogina (2004) step by step found the arithmetically complete logic of proofs and provability (which is now called GLA) in the union of the original languages of GL and LP.

Justification Logic

Plato: **Knowledge $\sim$ Justified True Belief**

Hintikka, et al: **F is known $\sim$ F holds in all possible situations.**

This simplified approach leaves Justification off the picture. It allowed to built an applicable formal theory of knowledge, but has had a number of deficiencies, e.g., the Logical Omniscience Problem, Gödel's Provability problem.

Justification Logic (Artemov, Fitting, Nogina, et al.):

$$t{:}F \sim \textit{t is an adequate evidence for F.}$$

Joining Implicit and evidence-based knowledge.

Artemov and Nogina (2004):

$$\mathsf{S4LP} = \mathsf{S4} \quad + \quad \mathsf{LP} \quad + \quad t{:}F \rightarrow \Box F \qquad (\text{or } t{:}F \rightarrow \Box t{:}F).$$

Multiple systems combining implicit knowledge '$\Box F$' and evidence-based knowledge '$t{:}F$'. A mathematical definition of Logical Omniscience via proof complexity and a series of results showing that in Justification Logic implicit knowledge is logically omniscient whereas evidence-based knowledge is not logically omniscient (S.A., Kuznets).

Epistemic models (Fitting, S.A., et al.)

Model is $(W, R, \mathcal{E}, \Vdash)$, where

- (W, R) is an S4-frame;

- $\mathcal{E}$ is an evidence function: for each term t and formula F, $\mathcal{E}(t, F)$ is a set of $u \in W$ where t is a possible evidence for F. An evidence function is monotonic

$$u \in \mathcal{E}(t, F) \text{ and } uRv \text{ yield } v \in \mathcal{E}(t, F)$$

and has natural closure properties that agree with operations of LP.

- $u \Vdash t{:}F$ iff $u \Vdash \Box F$ and $u \in \mathcal{E}(t, F)$.

Topological semantics for justifications - basic ideas

- based on Tarski's topological semantics $(\Box F)^* = \text{Interior}(F)^*$
- t's denote **measurements**; there is a measurement function $\mathcal{M}$ that for each t and F specifies an open set $\mathcal{M}(t, F)$ of 'possible outcomes' (not necessarily from F^*);
- $t{:}F \sim$ '*a set where measurement t confirms F*'. This reading is supported by the definition

$$(t{:}F)^* \; = \; \text{Interior}(F^*) \cap \mathcal{M}(t, F)$$

- first consider systems w/o operations on measurements.

System $\mathsf{S4B}_0$

$\mathsf{S4B}_0 \;=\; \mathsf{S4} \;+\; x{:}F \rightarrow F.$

In this system there are no any assumptions about measurements; they don't even necessarily produce open sets of outcomes. The definition of the measurement assertion is modified as follows:

$$(x{:}F)^* \;=\; F^* \cap \mathcal{M}(x,F)$$

System $\mathsf{S4B}_1$

$\mathsf{S4B}_1 \;=\; \mathsf{S4} \;+\; x{:}F \rightarrow \Box F.$

In this system measurement sets are still arbitrary (not necessarily open); however, the measurement assertions are interpreted normally

$$(x{:}F)^* \;=\; \text{Interior}(F^*) \cap \mathcal{M}(x, F)$$

System S4B$_2$

$$\mathsf{S4B}_2 \;=\; \mathsf{S4} \;+\; x{:}F \rightarrow F \;+\; x{:}F \rightarrow \Box x{:}F.$$

This system corresponds to the full operation-free version of S4LP. The measurement sets are open, the measurement assertions are interpreted normally

$$(x{:}F)^* \;=\; \mathrm{Interior}(F^*) \cap \mathcal{M}(x, F)$$

Introducing operations

A natural topological interpretation for the whole of S4LP.

Operation	Measurement Function
$t{:}(F \to G) \;\to\; (s{:}F \to (t\cdot s){:}G)$	$\mathcal{M}(s, F\to G) \cap \mathcal{M}(t, F) \subseteq \mathcal{M}(s\cdot t, G)$
$t{:}F \;\to\; !t{:}(t{:}F)$	$\mathcal{M}(t, F) \subseteq \mathcal{M}(!t, t{:}F)$
$s{:}F \vee t{:}F \to (s+t){:}F$	$\mathcal{M}(s, F) \cup \mathcal{M}(t, F) \subseteq \mathcal{M}(s+t, F)$

Conclusions

Justification logics can be provided with a topological semantics which reads justification assertions $t{:}F$ as

measurement t supports F.

This semantics is a natural extension of the Tarski topological semantics for the modal logic S4.

Future work

This is still work in progress. The next natural steps are to extend this interpretation to establish completeness for the real topology $\mathbb{R}^n$. It looks this can be accomplished by the existing technique.