Kurt Gödel Centenary:

His Legacy in Mathematics and Computer Science

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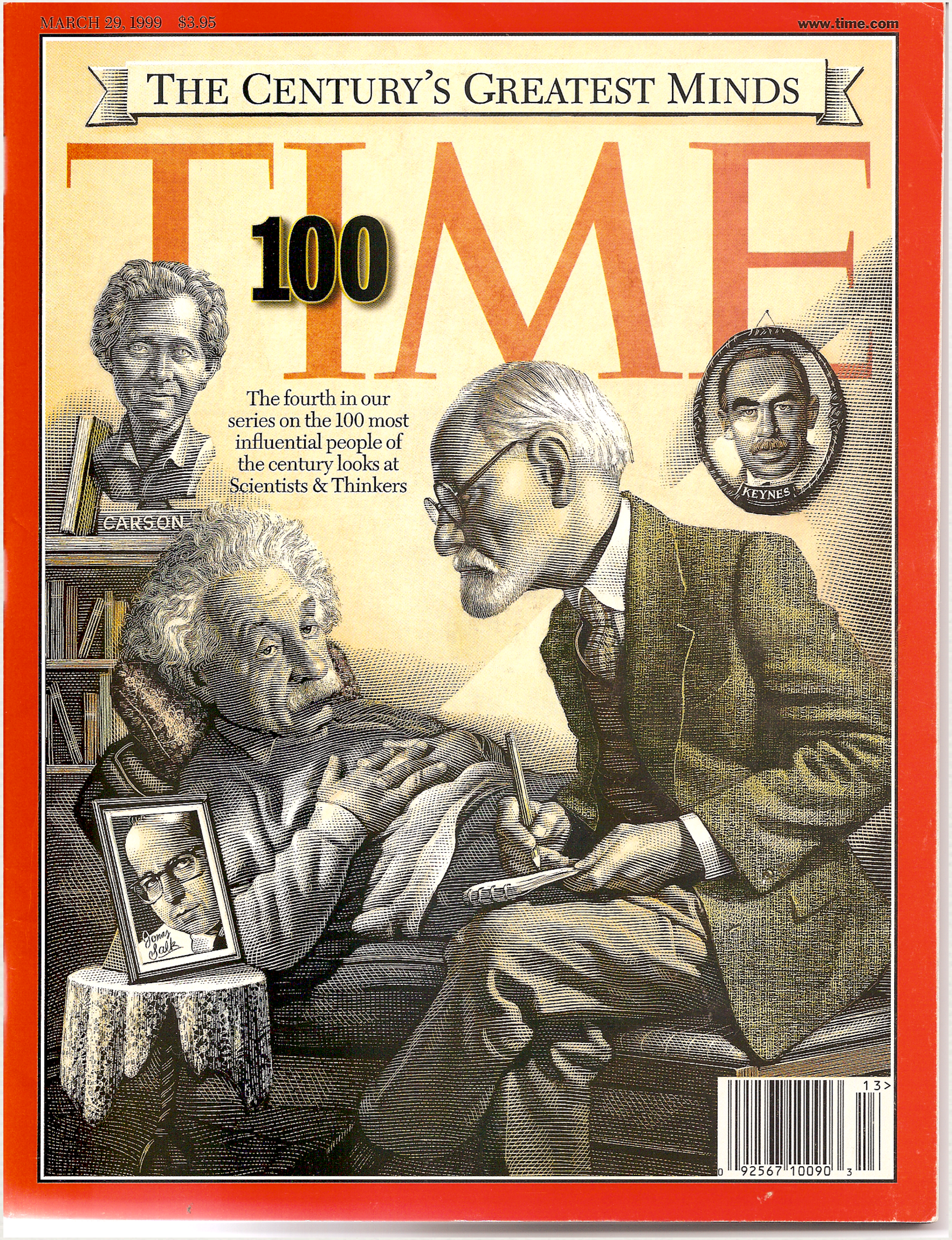
CUNY Computer Science Colloquium ,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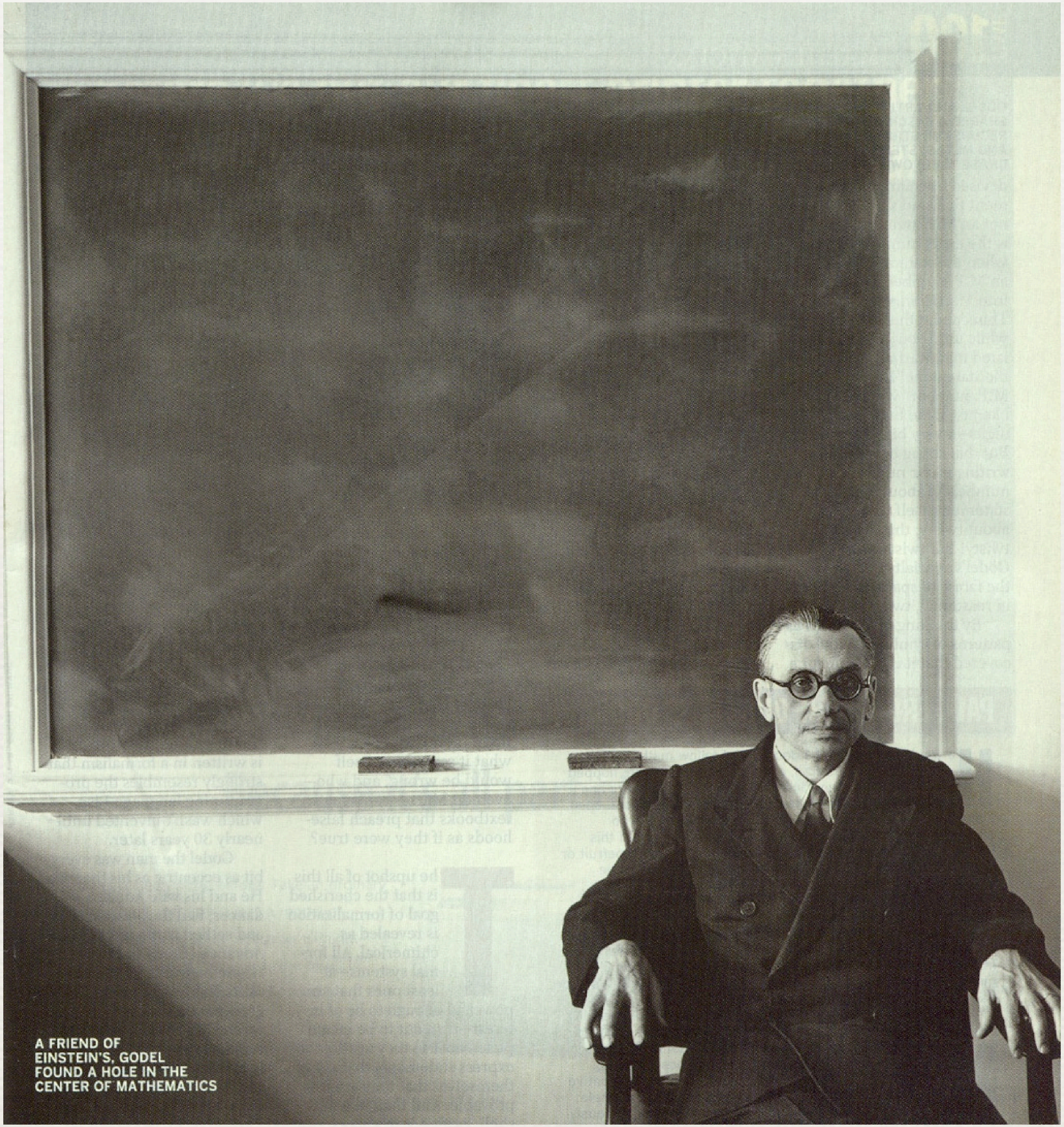
- Technology - 6 (airplane, rocket, TV, transistor, plastic, WWW)
- Biology & Medicine - 4 (psychoanalysis, penicillin, DNA, polio)
- Physics & Astronomy - 3 (Einstein, Fermi, Hubble)
- Anthropology - 1 (The Leakeys)
- Economy - 1 (Keynes)
- Environment - 1 (Rachel Carson)
- Psychology - 1 (Piaget)

- Computer Science - 1 (?)
- Mathematics - 1 (?)
- Philosophy - 1 (?)

Mathematics:

Kurt Gödel



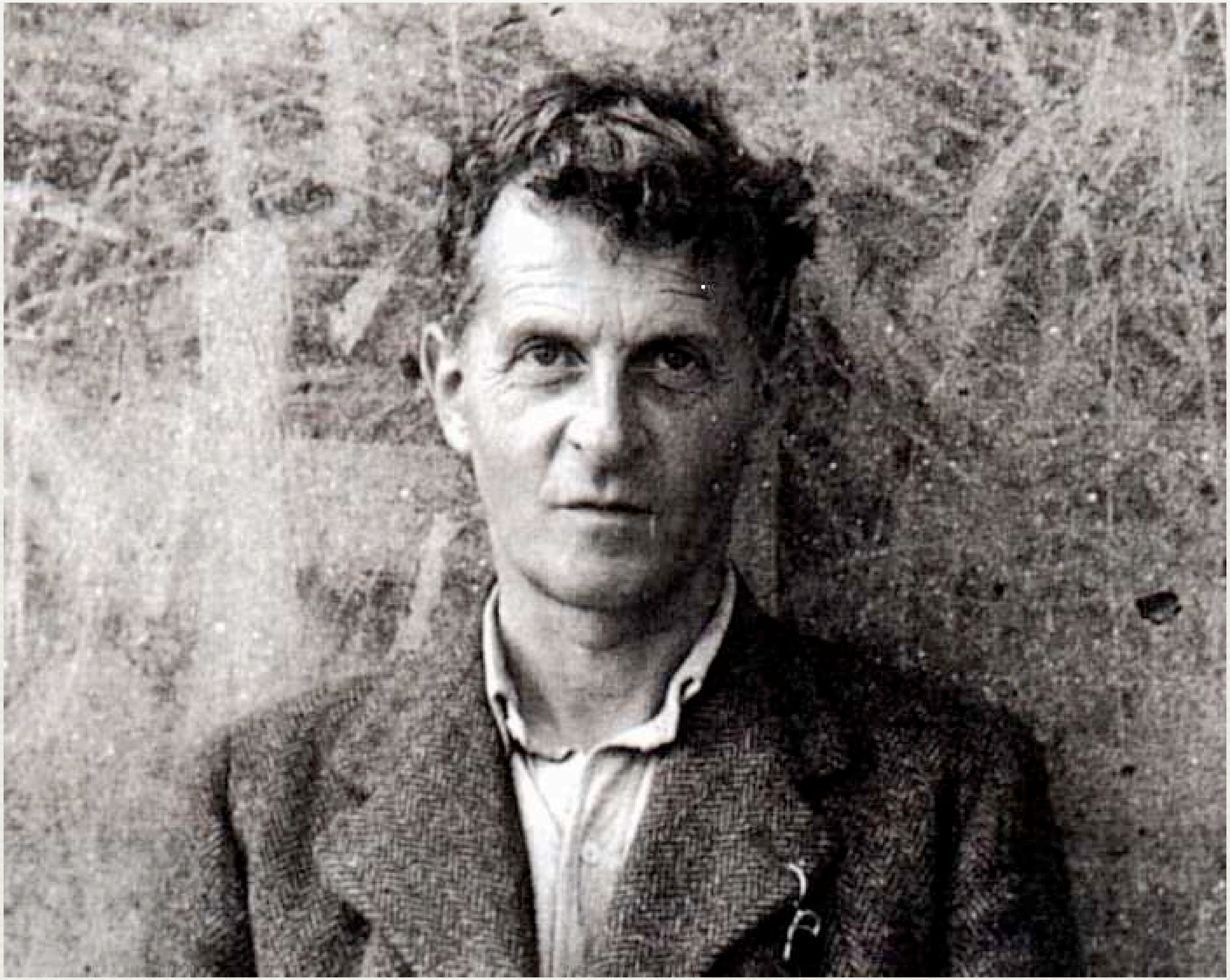
A FRIEND OF EINSTEIN'S, GODEL FOUND A HOLE IN THE CENTER OF MATHEMATICS

Computer Science:

Alan Turing

Philosophy:

Ludwig Wittgenstein



Logic, Logic and Logic

All three winners: Gödel, Turing and Wittgenstein began as mathematical logicians.

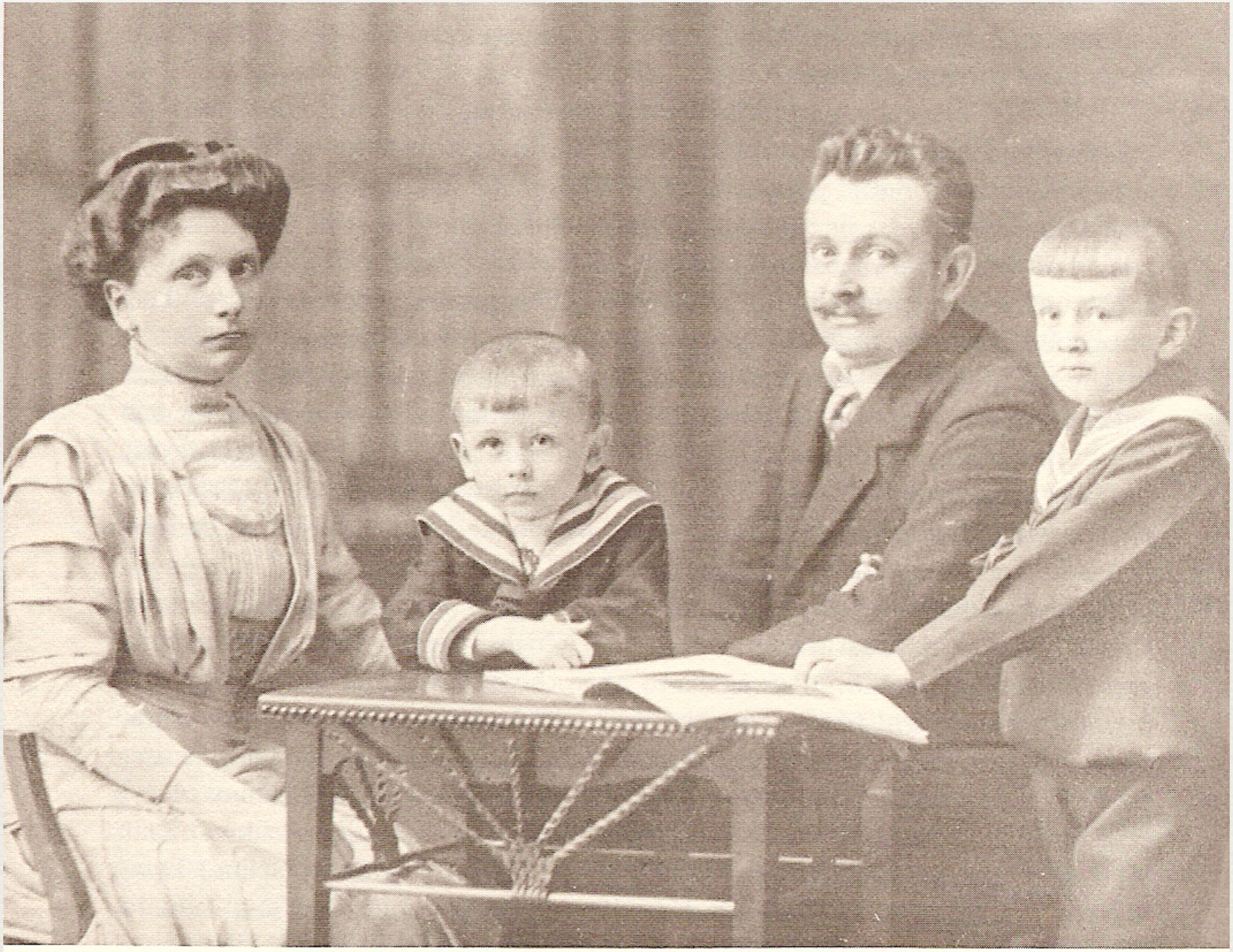
Logic is a truly interdisciplinary subject, which has a lot more to offer

Kurt Gödel (1906-1978)

Born on April 28, 1906 in Brno , then Austria-Hungary, now Czech Republic.

Wealthy parents (textile): father Rudolf, no academic education, mother Marianne, had a literary education and had undertaken part of her school studies in France.

Elder brother, Rudolf, who became a radiologist.



The Gödel family: Mother Marianne, son Kurt, father Rudolf, elder son Rudolf

Student years 1923-1929

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Entered the University of Vienna in 1923, first undecided between Mathematics and Physics.

Eventually got interested in Mathematical Logic and made his Ph.D. in 1929 under a famous mathematician Hans Hahn, who is best remembered for the Hahn-Banach theorem.

Hans Hahn (1873-1934)

Ph.D. Dissertation, 1929

The completeness of the first order logical calculus: **for the logical language with one sort of objects and quantifiers ∀ (for all), ∃ (there exists), over them, predicate and functional symbols, the standard system of axioms and rules of inference is sufficient for deriving all valid logical laws.**

Established adequacy of the axiomatic method for the task of capturing all valid laws of logic.


Dr Kurt Gödel

University of Vienna 1930-39

1931 - Incompleteness Theorems published

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I: ***In any sufficiently rich consistent axiomatic mathematical system there are propositions that cannot be proved or disproved within the axioms of the system.***

University of Vienna 1930-39

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I: ***In any sufficiently rich consistent axiomatic mathematical system there are propositions that cannot be proved or disproved within the axioms of the system.***

II: ***The consistency of such axiomatic system can be presented as a formal proposition, and such a proposition cannot be proved within the system itself.***

University of Vienna 1930-39

1932 - Habilitation, 1933 - Privatdozent

1932-1938 - papers on intuitionistic logic and modal logic of provability

1934 - first visit to the US: IAS

1938 - married Adele Porkert (divorced nightclub dancer, which was 6 years older than him) who was a great support to him for the rest of his life.

"Kurtele, if I compare your lecture with the others, there is no comparison"

-Adele Gödel

University of Vienna 1930-39

- 1938-39 - the second trip to the US: IAS and the University of Notre Dame
- 1938 - The announcement of the consistency of AC and GCH, in Proceedings of NAS
- 1938 - Anschluss: Austria becomes a part of Nazy Germany.
- 1939 (September) WWII begins, the Gödels apply for an immigrant visa to the US

Princeton 1940-1978

1940 - "Consistency of the axiom of choice and of the generalized continuum-hypothesis with the axioms of set theory"- monograph

1940 - Journey from Vienna to Princeton via Russia (trans-Siberian railroad), Japan, San Francisco, and then by train to the East Coast.

1946 - permanent member of IAS

1948 - US citizenship

1953 - full professor at IAS, emeritus since 1976

Friendship with Einstein

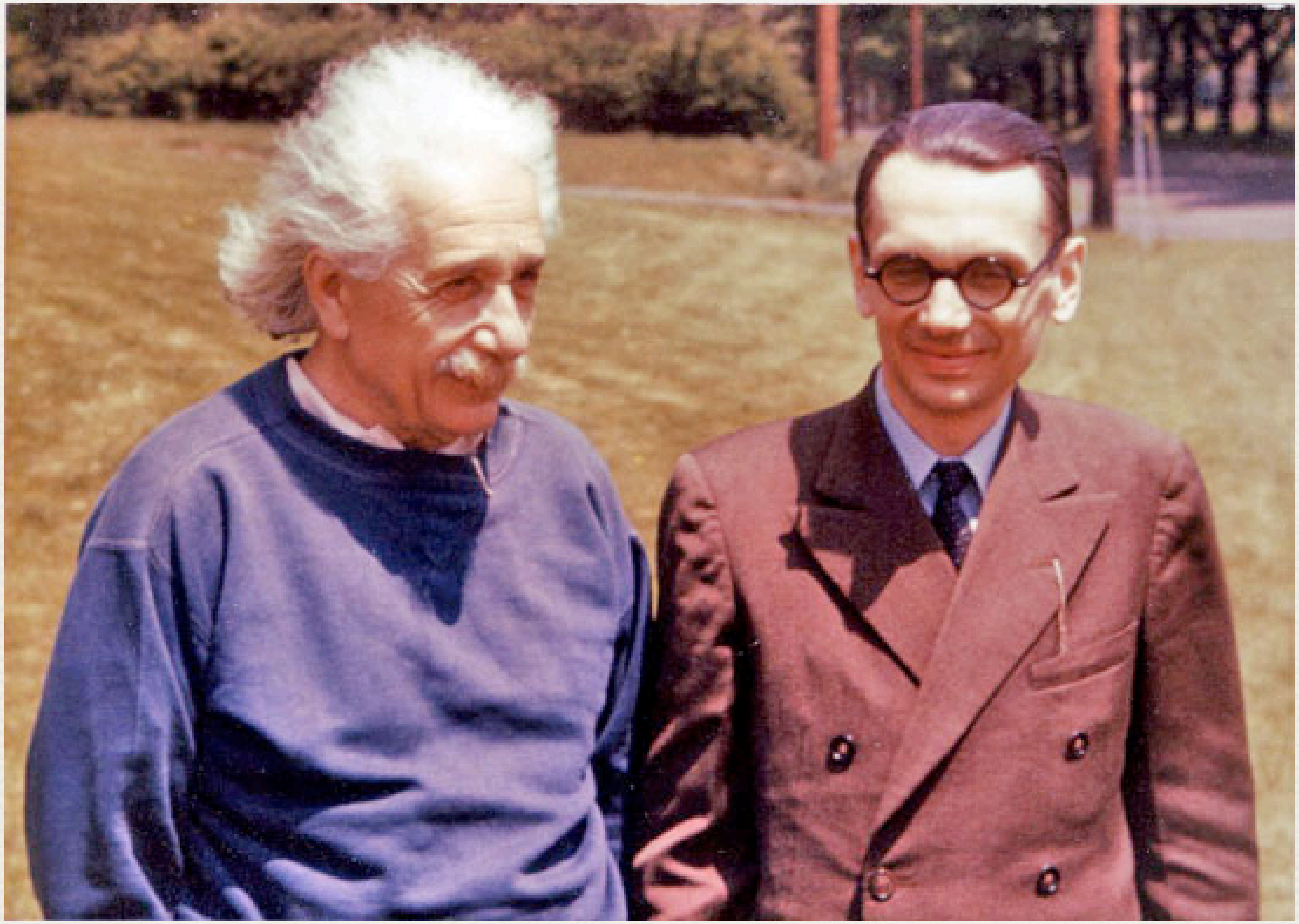


Photo taken on May 13, 1947

Friendship with Einstein

They were very different in almost every personal way - Einstein gregarious, happy, full of laughter and common sense, and Gödel extremely solemn, very serious, quite solitary, and distrustful of common sense as a means of arriving at the truth.

Friendship with Einstein

They were very different in almost every personal way - Einstein gregarious, happy, full of laughter and common sense, and Gödel extremely solemn, very serious, quite solitary, and distrustful of common sense as a means of arriving at the truth.
But they shared a fundamental quality: both went directly and wholeheartedly to the questions at the very center of things.

Ernst Straus, Reminiscences, 1982

Princeton 1940-1978

1940-1943 unsuccessful attempts to establish independence of AC and CH (accomplished by P.J. Cohen in 1963).

From 1943 on, Gödel devoted himself almost entirely to philosophy and metaphysics.

1947-1951 Unusual cosmological models that, in theory, permit "time travel."

1956 - In a letter to von Neumann: the first known formulation of "P versus NP" problem.

Princeton 1940-1978

1951 The first Einstein Award (with Julian Schwinger)

1951 Gibbs Lecturer by the AMS

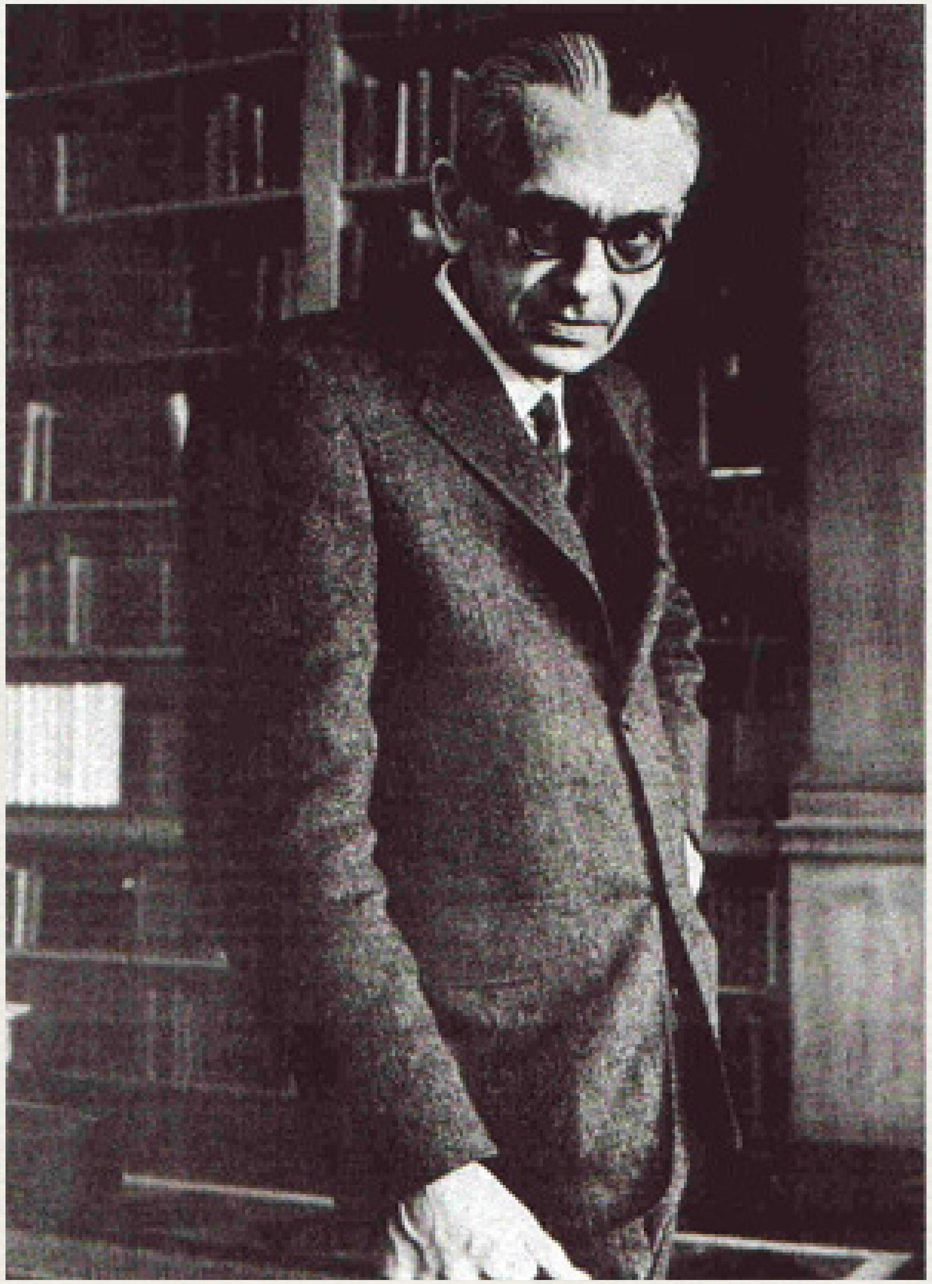
1955 Member of the NAS

1957 Member of the American Academy of Arts and Sciences

1968 Member of the Royal Society

1975 National Medal of Science

Princeton 1940-1978



Princeton 1940-1978

1977 July - Hospitalization of Adele Gödel for major surgery. Kurt Gödel became convinced that he was being poisoned and, refusing to eat to avoid being poisoned, essentially starved himself to death.

1978, January 14 - Kurt Gödel died sitting in a chair in his hospital room at Princeton. He weighed 65 pounds.

Incompleteness Theorem I

Axiom systems: efficient set of **axioms** and **rules of inference** - generator of **theorems.**

The desired properties:

Consistency - for no ***F***, ***F*** and ***not F*** are theorems

Soundness - all theorems are true

Completeness - for each meaningful ***F***, either ***F*** or ***not F*** is a theorem.

Note: Soundness yields Consistency, since for no ***F*** both ***F*** and ***not F*** can be true.

Incompleteness Theorem I

T is a sufficiently rich axiomatic system (capable of encoding elementary computations, the usual axioms for arithmetic "+" and "x" suffice).

The given proof of GI exploits Gödel's great invention - **internalization of computability**

Incompleteness Theorem I

Gödel I (Rosser's form):

T is consistent ⇒ T is incomplete

Incompleteness Theorem I

Gödel I (Rosser's form):

T is consistent ⇒ T is incomplete

We now establish a weaker version:

T is sound ⇒ T is incomplete

(i.e., there are always true but unprovable in T propositions)

Incompleteness Theorem I

Operating System (= Universal Machine)

$$U(p,n) = p(n)$$

any program (→ p) ***any input*** (→ n)

Compilers provide correct application, but cannot guarantee convergency. Hence, ***U(p,n)*** is **computable** but **not total,** e.g., for a syntactically correct nonsense program ***nonsense,***

U(nonsense,x) is undefined for each input ***x.***

Incompleteness Theorem I

Halting Problem:

"$U(i,i)$ **halts**" = "*program **i** terminates on input **i***"

is undecidable. O/w take a total computable function

$$f(n) = \begin{cases} U(n,n)+1, & \text{if } U(n,n) \text{ halts,} \\ 0 & \text{otherwise} \end{cases}$$

Pick such i that $U(i,\bullet) = f(\bullet)$.

Then $U(i,i) = f(i) = U(i,i) + 1$, a contradiction.

Incompleteness Theorem I

Weaker version: T **is sound** ⇒ T **is incomplete**

Proof. Suppose T is sound and complete, then there is a decision algorithm for the Halting Problem:

given i*, launch an algorithm that enumerates all theorems of* T *and wait until either* "$U(i,i)$ **halts**" *or* "$U(i,i)$ **does not halt**" *appears.*

By Completeness, the algorithm terminates.
By Soundness, the answer given is correct.

Incompleteness Theorem I

Profound consequences for the scope of axiomatic methods and Hilbert program:

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Profound consequences for the scope of axiomatic methods and Hilbert program:

- mathematics cannot be axiomatized completely
- even the theory of integer addition and multiplication (a.k.a. "arithmetic") cannot be completely axiomatized.
- Gödel's proof (quite different from the one above) provides a specific example of an independent sentence for each such T

Incompleteness Theorem I
Gödel's proof

For each *T,* Gödel builds arithmetic formulas

Proof(x,y) =

x* is a code of a proof of a formula having code *y

Provable(y) = ∃*x Proof(x,y)*

Internalization (Int):

For all *T, T* proves *F* ⇒ *T* proves ***Provable(*F*)***

For sound *T, T* proves *F* ⇔ *T* proves ***Provable(*F*)***

(***F*** is a proposition and **F** is a numeric code of ***F***)

Incompleteness Theorem I
Gödel's proof

Fixed Point Lemma: for any $A(x)$ there is a (fixed point) proposition F such that T proves

$$F \Leftrightarrow A(\mathrm{F})$$

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1. If T proves F, then
T proves $\textit{Provable}(\mathbf{F})$, by Int;
T proves *not F,* by FP

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Proposition asserting its own unprovability: T proves

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1. If T proves F, then T proves $\textit{Provable}(\mathbf{F})$, by Int; T proves $\textit{not } F$, by FP.

2. If T proves $\textit{not } F$, then T proves $\textit{Provable}(\mathbf{F})$; T proves F.

Incompleteness Theorem II

Gödel's famous consistency formula.

***Consistent(T)* - "*not Provable*(0=1)"**

GII: ***T* is consistent ⇒ *T* does not prove *Consistent(T)***

Gödel's proof: **Internalize GI(1)**

Incompleteness Theorem II
Gödel's proof

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GI(1): if *T* is consistent, then *T* does not prove *F*

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Gödel's proof

GI(1): if *T* is consistent, then *T* does not prove *F*

↓↓

INTERNALIZATION IN *T* (which Gödel never did)

↓↓

T proves ***Consistent(T)* ⇒ *not Provable(*F*)***

Incompleteness Theorem II
Gödel's proof

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Incompleteness Theorem II
Gödel's proof

GI(1): if T is consistent, then T does not prove F

$\downarrow\downarrow$

INTERNALIZATION IN ***T*** (which Gödel never did)

$\downarrow\downarrow$

T proves ***Consistent(T)*** $\Rightarrow$ ***not Provable(F)***

T proves ***Consistent(T)*** $\Rightarrow$ ***F***, by FP

if *T* is consistent, then *T* does not prove ***Consistent(T)***

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- Any consistency proof for T should necessarily use methods not internalizable in T.

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- In particular, if *T* is a theory that represents "all" methods of Mathematics, that the consistency of *T* cannot be established by conventional mathematical methods.

Incompleteness Theorem II

Dramatic consequences for the foundations:

- Any consistency proof for *T* should necessarily use methods not internalizable in *T*.
- In particular, if *T* is a theory that represents "all" methods of Mathematics, that the consistency of *T* cannot be established by conventional mathematical methods.
- Hilbert's Program of formalizing mathematics and proving it consistent by reliable "finitistic" methods cannot be fulfilled.

Mathematical Incompleteness

- Diophantine form (Matiyasevich)
- Combinatorial principles (Paris-Harrington)
- Sequences and Games (Paris-Kirby)
- Boolean Relation Theory (H. Friedman)

Diophantine Form

Matiyasevich (1970): Corresponding to any given axiomatization capturing arithmetic (e.g., Peano Arithmetic, Set Theory with whatever new principles added), one can explicitly construct a Diophantine equation which has no solutions, but such that this fact cannot be proved within the given axiomatization.

Known complexity bounds: $n=9$ or $\partial=4$

Combinatorial principles

Paris-Harrington (1977). Independence from PA of a modified form of the Finite Ramsey Theorem:

*For any positive integers n, k, m we can find N with the following property: if we color each of the n element subsets of {1, 2, 3,..., N} with one of k colors, then we can find a subset Y with at least m elements, such that all n element subsets of Y have the same color, **and the number of elements of Y is at least the smallest element of Y.***

Goodstein sequences

In the hereditary representation of integers *bump the base and subtract 1*:

$$G_0(265) = 265 = 2^{2^{2+1}} + 2^{2+1} + 1$$
$$G_1(265) = 3^{3^{3+1}} + 3^{3+1}$$
$$G_2(265) = 4^{4^{4+1}} + 4^{4+1} - 1 =$$
$$4^{4^{4+1}} + 3 \cdot 4^4 + 3 \cdot 4^3 + 3 \cdot 4^2 + 3 \cdot 4 + 3$$

Goodstein: $G_k(n)$ converges to 0 for any n

Paris - Kirby, 1982: independent of PA

Gödel and Computer Science

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- Primitive recursive functions, internalization of computations and proofs, fixed point theorem

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Gödel and Computer Science

- Primitive recursive functions, internalization of computations and proofs, fixed point theorem
- Completeness Theorem - Logic Programming
- Formulation of the prominent “P vs NP” problem in a letter to von Neumann
- Constructive semantics and proof mining. Higher order typed languages.
- Epistemic modal logics and the first draft of justification logics

“P vs NP” problem

Now one of the Millennium $1M problems in Math

Fast (polynomial) verification of a lucky guess solution = **Nondeterministic Polynomial**

vs

Finding a solution (≠? **Polynomial**)

Gödel in 1956 formulated this problem in logical terms of the length of proofs and the number of steps needed to find a proof.

Godel's Calculus for Provability

$\Box A \approx A$ ***is provable***

- Classical propositional logic
- $\Box(A \rightarrow B)$ & $\Box A. \;\rightarrow\; \Box B$
- $\Box A \rightarrow A$
- $\Box A \rightarrow \Box\Box A$
- If F is derived, then $\Box F$ is derived also.

Led to a vast industry of the Logic of Knowledge

Logic of Proofs

Anticipated by Gödel in 1938, found by S.A. in 1995

A complete logic with additional atoms

t:F ≈ t is a proof of F

Solved problems left open by Gödel and Kolmogorov

Evolved into a formal theory of justification, belief and knowledge. Captures classical Plato's approach to knowledge via justification. Now known as

Justification Logic

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