Logic of knowledge with justifications from the provability perspective.

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Abstract

An issue of a logic of knowledge with justifications has been discussed since the early 1990s. Such a logic along with the usual knowledge operator $\Box F$ *"F is known"* should contain assertions $t{:}F$ *"t is an evidence of F"*. In this paper we build two systems of logic of knowledge with justifications: LPS4, which is an extension of the basic epistemic logic S4 by an appropriate calculus of evidences corresponding to the logic of proofs LP together with the principle that justification implies knowledge, and $\mathsf{LPS4}^-$, which is LPS4 augmented by the mixed implicit/explicit negative introspection principle. We offer a provability semantics for LPS4 and $\mathsf{LPS4}^-$ where the epistemic modality $\Box F$ is interpreted as *"F is true and provable"* and the evidence assertions $t{:}F$ as *"t is a proof of F"*. We find Kripke semantics and establish a number of fundamental properties of LPS4 and $\mathsf{LPS4}^-$. On the way to those systems we find the minimal joint logic of proofs and formal provability, LPGL, complete with respect to the standard provability semantics.

1 Introduction.

A need for a logic of knowledge with justifications has been discussed in [6]. Such a logic along with the usual knowledge operators $\Box F$ *"F is known"* should contain assertions $t{:}F$ *"t is an evidence of F"*, which brings explicit and quantitative components to the logic of knowledge.

The explicit character of judgments significantly expands the expressive power of epistemic logics. The original epistemic modality $\Box F$ should be regarded as "potential knowledge", or "knowability"[1] rather than actual knowledge. Evidence operators $t{:}F$ model real knowledge of the agent which provides a justification that F true in all situations. This intuition is consistent with a Kripke style semantics of the logic of knowledge with justifications developed below. An assertion $t{:}F$ if true in a world holds in all worlds of a model, the present, future, or past ones.

The quantitative component of justification terms could be useful in dealing with the well-known *logical omniscience problem* [10, 22, 25, 26] since an evidence term t in $t{:}F$ carries information about how hard it was to justify F from given assumptions.

In this paper we deduce basic principles of knowledge and justifications from the laws of provability. The provability semantics is a representative special case of epistemic reading of modal logic and as such shreds light on the general logic of knowledge with justifications.

Provability interpretation of modal logic where $\Box F$ is interpreted as

$$F \textit{ is provable in Peano Arithmetic.} \tag{1}$$

was first considered by Gödel in [13] and actively studied since Solovay's paper [30] of 1976 (cf. [7, 8, 16, 29]). In this paper we use the provability logic GL as a source of a solid provability semantics for the logic of knowledge with justifications. Provability logic GL is not compatible with epistemic logic S4, mainly because arithmetical provability is not reflexive. However, S4 can be modelled in GL by using so called *strong provability operator*

$$F \textit{ is true and provable in Peano Arithmetic.} \tag{2}$$

S4 is sound with respect to the strong provability semantics, the extension S4Grz of S4 by Grzegorcsyk schema $\Box(\Box(F \rightarrow \Box F) \rightarrow F) \rightarrow F$ provides a complete propositional axiomatization of strong provability [7, 15, 18, 19]. Kripke models corresponding to S4Grz have S4-frames which do not distinguish possible worlds mutually accessible from each other.

The idea of the logic of proofs as an explicit counterpart of S4 first appeared in Gödel's [14]. The formal system LP of the logic of proofs was introduced in [2, 3]. LP describes all valid principles of proof operators $t{:}F$

$$t \textit{ is a proof of } F \textit{ in Peano Arithmetic} \tag{3}$$

with an appropriate set of operations on proofs sufficient to realize modal logic S4 explicitly [3]. A similar explicit counterpart of S5 was found in [5]. A promising semantical approach to the logic of proofs as a general calculus of evidences in the epistemic framework has been developed in [12, 21].

Joint logics of proofs and provability studied in [1, 23, 24, 27, 28] are of special interest for this paper since they serve as a prototype of the logic of knowledge with evidences. The first system B of provability and explicit proofs without operations on proof terms was found, supplied with Kripke semantics and shown

[1] Cf. Fitting's paper [12].

to be arithmetically complete in [1]. Arithmetically complete system BGrz of strong provability and proofs without operations was introduced in [23, 24] (a comprehensive system of propositional operations on proofs was not known till [2]). Finally, in [27, 28] a system LPP of provability and explicit proofs containing both LP and GL was found in the language with some additional operations.

In this paper we pursue two main goals:

A) to axiomatize the arithmetically complete system, LPGL, of proofs and provability in the joint language of LP and GL;

B) to introduce logics of knowledge with justifications based on the principles from LPGL and thus enjoying the standard provability semantics.

In section 2 we introduce the logic LPGL of proofs and formal provability which is the minimal arithmetically complete extension of LP and GL. We establish Kripke and arithmetical completeness as well as a decidability of LPGL for a given constant specification. LPGL is a refinement of the system LPP by Sidon-Yavorskaya [27, 28].

In section 3 we construct two systems of logic of knowledge with justifications. The a basic one, LPS4 consists of S4 combined with LP as a calculus of evidences and a principle connecting implicit and explicit knowledge operators: $t{:}F \rightarrow \Box F$. The system LPS4 may be regarded as the basic logic of knowledge with justifications when no specific assumptions are made concerning the behavior of explicit knowledge. LPS4 enjoys the arithmetical provability semantics when $\Box F$ is interpreted as the strong provability (2) and the evidence judgments $t{:}F$ are interpreted as proof assertions (3). LPS4 is sound with respect to Fitting models (cf. [12]) and has an important internalization property typical for logics with rich enough system of terms for explicit knowledge.

The other system, LPS4$^-$, is LPS4 augmented by the principle of *negative introspection* $\neg(t{:}F) \rightarrow \Box\neg(t{:}F)$ (which explains the superscript "minus" in the name of the system). Alternatively, LPS4$^-$ can be axiomatized over S4 plus LP by the principle of *decidability of evidences* $\Box t{:}F \vee \Box\neg(t{:}F)$. We show that LPS4$^-$ enjoys the standard provability semantics, is decidable and complete with respect to a natural Kripke style semantics. In order to get a complete system in the language of LPS4$^-$, one has to add Grzegorczyk schema to LPS4$^-$.

We believe that systems LPS4 and LPS4$^-$ provide a proper framework for reasoning about knowledge and enables us to express principles which could find its formulation neither in the pure modal language nor in the pure language of proof terms. For example, the modal principle of negative introspection $\neg\Box F \rightarrow \Box\neg\Box F$ is valid neither in the provability semantics nor in the strong provability semantics. A purely explicit version of negative introspection $\neg(x{:}F) \rightarrow t(x){:}\neg(x{:}F)$ does not hold in the logic of proofs LP neither. However, in the logic of knowledge with justifications LPS4$^-$ the negative introspection appears in a synthetic explicit-implicit form $\neg(t{:}F) \rightarrow \Box\neg(t{:}F)$ valid in both provability and strong provability semantics, which provides a good reason for accepting this principle in general.

2 Joint logic of proofs and formal provability.

There are several motivations to study the joint logic of proofs and formal provability LPGL below. On the foundational side, we wanted to find an arithmetically complete closure of the provability logic GL with atoms $\Box F$, and the logic of proofs LP with atoms $t{:}F$. In addition to principles imported from GL and LP such a closure contains provability principles in a joint language of explicit proofs and formal provability which are dictated by the underlying provability models: *connection* $t{:}F \rightarrow \Box F$, *negative introspection* $\neg t{:}F \rightarrow \Box\neg(t{:}F)$ or *weak reflexivity* $t{:}\Box F \rightarrow F$. In section 3 we use these provability principles to build logics of explicit knowledge.

2.1 Formulation and basic properties of LPGL

Definition 1. *Proof polynomials* for LPGL are terms built from *proof variables* $x, y, z, \ldots$ and *proof constants* $a, b, c, \ldots$ by means of three operations, *application* "$\cdot$" (binary), *union* "+" (binary), and *proof checker* "!" (unary).

Though proof polynomials for LPGL have the same set of operations as the ones for the logic of proof LP, there are more axioms and hence more choices to specify proof constants in LPGL, which makes the latter more expressive than the standard LP-polynomials. Note, however, that LPGL-polynomials extend LP-polynomials in the minimal possible way, i.e. by adding only proof constants corresponding to additional axioms.

Definition 2. Using t to stand for any proof polynomial and S for any sentence variable, the formulas are defined by the grammar

$$A = S \mid A_1 \rightarrow A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \neg A \mid \Box A \mid t{:}A.$$

Subformulas *Sub(F)* of a given formula F are defined as usual, with the extra clause

$$Sub(t{:}A) = t{:}A \cup Sub(A).$$

We assume also that "$t{:}$", "$\Box$" and "$\neg$" bind stronger than "$\wedge, \vee$", which bind stronger than "$\rightarrow$".

Definition 3. The *logic of proofs and formal provability*, LPGL, has axioms of both LP and GL, three specific principles connecting explicit proofs with formal provability and rules *Modus Ponens* and *Constant Specification* as shown below.

I. **Classical propositional logic**

The standard set of axioms A1-A10 from [17] (or a similar system)
R1. *Modus Ponens*

II. **Provability Logic GL**

GL1. $\Box(F \rightarrow G) \rightarrow (\Box F \rightarrow \Box G)$ (*implicit application*)

GL2. $\Box F \to \Box\Box F$ (*implicit proof checker*)
GL3. $\Box(\Box F \to F) \to \Box F$ (*Löb schema*)
R2. $\vdash F \;\Rightarrow \vdash \Box F$ (*necessitation rule*)
R3. $\vdash \Box F \Rightarrow \vdash F$ (*reflexivity rule*)

III. **Logic of Proofs** LP

LP1. $s{:}(F \to G) \to (t{:}F \to (s{\cdot}t){:}G)$ (*application*)
LP2. $t{:}F \to !t{:}(t{:}F)$ (*proof checker*)
LP3. $s{:}F \to (s{+}t){:}F, \quad t{:}F \to (s{+}t){:}F$ (*union*)
LP4. $t{:}F \to F$ (*explicit reflexivity*)
R4. $\vdash c{:}A$, where A is an axiom from I - IV and c is a proof constant (*constant specification rule*)

IV. **Principles connecting explicit and formal provability**

C1. $t{:}F \to \Box F$ (*explicit-implicit connection*)
C2. $\neg(t{:}F) \to \Box\neg(t{:}F)$ (*negative introspection*)
C3. $t{:}\Box F \to F$ (*weak reflexivity*)

Naturally, all axioms and rules are applied across sections I-IV. LPGL is closed under substitutions of proof polynomials for proof variables and formulas for propositional variables, since all axioms and rules are invariant with respect to those substitutions. LPGL contains both LP and GL, enjoys the deduction theorem $\Gamma, F \vdash G \;\Rightarrow\; \Gamma \vdash F \to G$. The standard proof by induction on a derivation of G from Γ, F fits here with minor modifications.

Definition 4. *Constant specification CS* is a finite set $\{c_1{:}A_1, \ldots, c_n{:}A_n\}$ of formulas, where each A_i is an axiom from I-IV and each c_i is a proof constant. By default, with each derivation in LPGL we associate a constant specification *CS* introduced in this derivation by the rule of *constant specification.* By LPGL_{CS} we mean a subsystem of LPGL where the rule of *constant specification* is restricted to producing formulas from a given *CS* only. In particular, $\mathsf{LPGL}_{\emptyset}$ is a subsystem of LPGL without any constant specifications.

The way of using proof constants in LPGL derivations is typical for the logic of proofs. Whenever we need a proof term for a given axiom A, we introduce a constant specification $c{:}A$. When claiming that F is derivable in LPGL we mean a derivation with a constant specification *CS* associated with this derivation:

$$F \textit{ is derivable given } c_1{:}A_1, \ldots, c_n{:}A_n.$$

Comment 1. The reflexivity rule R3 is usually omitted in the standard formulation of GL since it is an admissible rule of the latter (cf. [7, 8]). The same holds here: the rule R3 is derivable from the rest of LPGL (lemma 6). However, we need R3 as a postulated rule of the system to guarantee a good behavior of the fragments of LPGL corresponding to specific constant specifications. Another curious feature of the system is the fact that the weak reflexivity axiom C3 $t{:}\Box F \to F$ is derivable from the rest of $\mathsf{LPGL}_{\emptyset}$.

Lemma 1. $\mathsf{LPGL}_\emptyset \vdash t{:}\Box F \to F$.

Proof.

1. $\neg\Box F \to \neg t{:}\Box F$ (contrapositive of LP4);
2. $\neg t{:}\Box F \to \Box(\neg t{:}\Box F)$ axiom C2;
3. $\Box(\neg t{:}\Box F) \to \Box(t{:}\Box F \to F)$, by reasoning in GL;
4. $\neg\Box F \to \Box(t{:}\Box F \to F)$, from 1, 2 and 3;
5. $\Box F \to \Box(t{:}\Box F \to F)$, by reasoning in GL;
6. $\Box(t{:}\Box F \to F)$, from 4 and 5;
7. $t{:}\Box F \to F$, by R3. $\Box$

However, proof constants corresponding to the weak reflexivity axiom C3 are needed to guarantee the internalization property of LPGL (Proposition 2). Hence, we keep C3 as a basic postulate of the system to conveniently place it under the scope of the constant specification rule R4.

There are other innocent redundancies in the above formulation of LPGL, e.g. GL2 is derivable from the rest of the system [8]. More examples are given by corollary 1 and lemma 7.

Lemma 2. *The following are provable in* $\mathsf{LPGL}_\emptyset$ *(hence in* LPGL *and in* LPGL_{CS} *for any constant specification CS).*

1. $x{:}F \to \Box x{:}F$ *(positive introspection)*
2. $\Box x{:}F \vee \Box\neg x{:}F$ *(decidability of proof assertions)*

Proof.

1. $x{:}F \to !x{:}x{:}F$, by LP2,
   $!x{:}x{:}F \to \Box x{:}F$, by C1
   $x{:}F \to \Box x{:}F$, by propositional logic

2. $x{:}F \to \Box x{:}F$, by the previous item of this lemma
   $\neg(x{:}F) \to \Box\neg(x{:}F)$, by C2 $\Box$
   $\Box x{:}F \vee \Box\neg x{:}F$, by propositional logic

Lemma 3. LPGL_{CS} *proves* $F \quad \Leftrightarrow \quad \mathsf{LPGL}_\emptyset$ *proves* $\bigwedge \mathrm{CS} \to F$.

Proof. Similar to Lemma 2.1 from [28], by induction on a derivation of F in LPGL_{CS}. The only nontrivial cases are the rules of necessitation and reflection.

If F is obtained by the necessitation rule, i.e. F is $\Box G$ and $\mathsf{LPGL}_{CS} \vdash G$, then, by the induction hypothesis, $\mathsf{LPGL}_\emptyset \vdash \bigwedge CS \to G$. By GL reasoning,

$$\mathsf{LPGL}_\emptyset \vdash \Box \bigwedge CS \to \Box G.$$

By positive introspection (Lemma 2.1) and some trivial GL reasoning,

$$\mathsf{LPGL}_\emptyset \vdash \bigwedge CS \to \Box \bigwedge CS,$$

hence $\mathsf{LPGL}_\emptyset \vdash \bigwedge CS \to F$.

If F is obtained by the reflection rule, then $\mathsf{LPGL}_{CS} \vdash \Box F$ and, by the induction hypothesis, $\mathsf{LPGL}_\emptyset \vdash \bigwedge CS \to \Box F$, hence $\mathsf{LPGL}_\emptyset \vdash \neg \bigwedge CS \vee \Box F$. By the negative introspection (axiom C2) and some GL reasoning,

$$\mathsf{LPGL}_\emptyset \vdash \neg \bigwedge CS \to \Box \neg \bigwedge CS.$$

Therefore, $\mathsf{LPGL}_\emptyset \vdash \Box \neg \bigwedge CS \vee \Box F$ and $\mathsf{LPGL}_\emptyset \vdash \Box(\neg \bigwedge CS \vee F)$. By the reflection rule, $\mathsf{LPGL}_\emptyset \vdash \neg \bigwedge CS \vee F$, hence $\mathsf{LPGL}_\emptyset \vdash \bigwedge CS \to F$. $\Box$

Note, that both positive and negative introspection are needed to reduce the whole of LPGL to its fragment of unspecified constants $\mathsf{LPGL}_\emptyset$ and LPP to $\mathsf{LPP}_\emptyset$.

Lemma 4. *For any formula F there are proof polynomials $\mathtt{up}_F(x)$ and $\mathtt{down}_F(x)$ such that* LPGL *proves*

1. $x{:}F \to \mathtt{up}_F(x){:}\Box F$
2. $x{:}\Box F \to \mathtt{down}_F(x){:}F$

Proof.

1.	$x{:}F \to \Box F$,	by C1
	$a{:}(x{:}F \to \Box F)$,	specifying constant a, by *constant specification rule*
	$!x{:}x{:}F \to (a\cdot !x){:}\Box F$,	by LP1 and propositional logic
	$x{:}F \to !x{:}x{:}F$,	by LP2
	$x{:}F \to (a\cdot !x){:}\Box F$,	by propositional logic

It suffices now to put $\mathtt{up}_F(x)$ equal to $a\cdot !x$ such that $a{:}(x{:}F \to \Box F)$.

2.	$x{:}\Box F \to F$,	by C3
	$b{:}(x{:}\Box F \to F)$,	specifying constant b, by *constant specification rule*
	$!x{:}x{:}\Box F \to (b\cdot !x){:}F$,	by LP1 and propositional logic
	$x{:}\Box F \to !x{:}x{:}\Box F$,	by LP2
	$x{:}\Box F \to (b\cdot !x){:}F$,	by propositional logic

It suffices now to put $\mathtt{down}_F(x)$ equal to $b\cdot !x$ such that $b{:}(x{:}\Box F \to F)$ $\Box$

Proposition 1. (Constructive necessitation in LPGL)
If LPGL *proves F then* LPGL *proves $p{:}F$ for some proof polynomial p.*

Proof. Induction on a derivation of F. Base: F is an axiom. Then use constant specification rule. In this case p is an arbitrary proof constant. Induction step. If F is obtained from $X \to F$ and X by *Modus Ponens*. By the induction hypothesis, $\vdash s{:}(X \to F)$ and $\vdash t{:}X$, hence, by LP1, $\vdash (s\cdot t){:}F$, hence p is $s\cdot t$. If F is obtained by Necessitation, then $F = \Box G$ and $\vdash G$. By the induction hypothesis, $\vdash t{:}G$ for some proof polynomial t. Use lemma 4.1 to conclude that $\vdash \mathtt{up}_G(t){:}\Box G$ and put $p = \mathtt{up}_G(t)$. If F is obtained by the reflexivity rule R3, then $\vdash \Box F$. By the induction hypothesis, $\vdash t{:}\Box F$ for some proof polynomial t. Use lemma 4.2 to conclude that $\vdash \mathtt{down}_F(t){:}F$ and put $p = \mathtt{down}_F(t)$. If F is obtained by the constant specification rule, then F is $c{:}A$ for some constant c and axioms A. Use the proof checker axiom LP2 to derive $!c{:}c{:}A$, i.e. $!c{:}F$. Here p is $!c$. Note that proof polynomial p is ground and built from proof constants by applications and proof checker operations only. Moreover, the presented derivation of $p{:}F$ uses neither rule R2 nor rule R3. $\Box$

Corollary 1. *The necessitation rule R2 is derivable from the rest of* LPGL.

Indeed, if $\vdash F$ then, by proposition 1, $\vdash p{:}F$ for some proof polynomial p. By C1, $\vdash \Box F$.

Note that the rule of necessitation is not redundant in LPGL_{CS} for any constant specification CS. Indeed, to emulate the rule of necessitation one needs to apply constructive necessitation to the unbounded set of theorems of LPGL_{CS}, which requires an unbounded set of constant specifications.

The following property of LPGL is a generalization on constructive necessitation (proposition 1). It is the explicit analogue of the modal logic rule, which holds in normal modal logics containing K4, e.g. S4, S5, GL and LPGL

$$\frac{A_1,\ldots,A_k,\Box B_1,\ldots,\Box B_n \vdash F}{\Box A_1,\ldots,\Box A_k,\Box B_1,\ldots,\Box B_n \vdash \Box F}$$

Proposition 2. (Lifting) *If $A_1,\ldots,A_k,y_1{:}B_1,\ldots,y_n{:}B_n \vdash F$ then for some proof polynomial $p(x_1,\ldots,x_k,y_1,\ldots,y_n)$*

$$x_1{:}A_1,\ldots,x_k{:}A_k,y_1{:}B_1,\ldots,y_n{:}B_n \vdash p(x_1,\ldots,x_k,y_1,\ldots,y_n){:}F.$$

Proof. Similar to proposition 1 with two new base clauses. If F is A_i, then x_i can be taken as p. If F is $y_j{:}B$, then p is equal to $!y_j$. $\Box$

Lemma 5. (Internalization property of LPGL) *If $A_1,\ldots,A_k \vdash F$ then for some proof polynomial $p(x_1,\ldots,x_k)$*

$$x_1{:}A_1,\ldots,x_k{:}A_k \vdash p(x_1,\ldots,x_k){:}F.$$

Proof. A special case of proposition 2. $\Box$

The internalization property states that any derivation in LPGL can be internalized and proof checked as a proof term in LPGL itself. Since LPGL extends typed combinatory logic (hence typed λ-calculus) one could compare the scopes of the internalization property and the Curry-Howard isomorphism. It is easy to see that the latter is a very special case of the former when $A_1,\ldots,A_k,F$ contain neither modalities nor proof polynomials.

Lemma 6. *The reflection rule R3 is derivable from the rest of* LPGL.

Proof. Suppose $\vdash \Box F$. By proposition 1, $\vdash p{:}\Box F$ for some proof polynomial p. By C3, $\vdash F$. $\Box$

Lemma 7. *Explicit reflection axiom $t{:}F \to F$ is derivable from the rest of* LPGL.

Proof. LPGL derives $t{:}F \to \mathtt{up}_F(t){:}\Box F$ without using explicit reflection axiom (cf. lemma 4.1). Apply C3: $\mathtt{up}_F(t){:}\Box F \to F$ to get the desired $t{:}F \to F$. $\Box$

2.2 Kripke models for LPGL

An LPGL-*model* in short is a GL-model where all other axioms of LPGL hold. In building the Kripke models for LPGL we closely follow the corresponding constructions from [1, 28], where all the ground work has been done carefully. On this basis we will give a schematic description of LPGL-models and refer the reader to [1, 28] for further details.

Definition 5. An LPGL-*model* is a triple $(K, \prec, \Vdash)$ where $(K, \prec)$ is a finite irreflexive tree, $\Vdash$ is a forcing relation between nodes of K and LPGL-formulas satisfying the following *forcing conditions.*

1. usual modal conditions for $\Box$, i.e. $\Vdash$ respects boolean connectives at each node, $a \Vdash \Box F$ iff $b \Vdash F$ for all $b \succ a$;
2. *stability* for every formula $t{:}F$ either all nodes of K force $t{:}F$ or all nodes of K force $\neg t{:}F$;
3. *reflexivity of explicit knowledge*: $a \Vdash t{:}F$ yields $a \Vdash F$;
4. if $a \Vdash s{:}(F \to G)$ and $a \Vdash t{:}F$ then $a \Vdash (s \cdot t){:}G$;
5. if $a \Vdash t{:}F$ then $a \Vdash !t{:}(t{:}F)$;
6. if $a \Vdash s{:}F$ then a forces both $(s+t){:}F$ and $(t+s){:}F$.

A formula F holds in a model M if F is forced at each node of M. The root node of a model is called *root*. Put $H(F) = \{\Box G \to G \mid \Box G \textit{ is a subformula of } F\}$. We call a model F-sound, if $root \Vdash H(F)$. Similar conditions on a Kripke model could be found in [1, 28]. Moreover, each LPP model is also a LPGL model. A model M is a *CS-model*, for a given constant specification $CS = \{c_1{:}A_1, c_2{:}A_2 \dots c_n{:}A_n\}$, if M is X-sound and X holds in M for each $X = c_i{:}A_i$ from this *CS*. A formula F is *CS-valid* if it holds in each F-sound CS-model.

Theorem 1. (Soundness) *If F is derivable in LPGL_{CS}, then F is CS-valid.*

Proof. The standard induction on derivations in LPGL. The only nontrivial case is the induction step corresponding to the reflection rule $\vdash \Box F \Rightarrow \vdash F$. By the induction hypothesis, $\Box F$ holds in each $\Box F$-sound CS-model. Suppose there is an F-sound CS-model M where F does not hold at a certain node a. There are two possibilities.

Case 1. a is not *root*. Then $root \nVdash \Box F$, hence $root \Vdash \Box F \to F$ and M is an $\Box F$-sound CS-countermodel for $\Box F$.

Case 2. a is *root*. Build a new model M' by adding to M a new node b below a and defining

$$b \Vdash X \quad \Leftrightarrow \quad a \Vdash X$$

for all X which are sentence variables of formulas $t{:}Y$. It is easy to see that the same equivalence holds for all X's which are subformulas of $\bigwedge CS \rightarrow F$. Indeed, the cases of boolean connectives are trivial. If X is $\Box Z$ and $b \Vdash X$, then, apparently, $a \Vdash X$. If $a \Vdash X$, then $a \Vdash \Box Z$, hence $a \Vdash Z$, since M is F-sound. By the definition of forcing, $b \Vdash \Box Z$ as well. Since $a \nVdash F$, $b \nVdash \Box F$ and M' is an $\Box F$-sound CS-countermodel for $\Box F$. □

Theorem 2. (Kripke completeness of LPGL_{CS}.)
If F is not derivable in LPGL_{CS}, then F is not CS-valid.

Proof. Follows from the Kripke completeness of LPP from [28]. This also establishes the decidability of LPGL_{CS} for any (finite) *CS*. □

2.3 Provability semantics for LPGL.

The provability semantics for LPGL in Peano Arithmetic PA is the natural blend of those for the provability logic GL and the logic of proofs LP. As one might expect, $\Box F$ is interpreted as *there is a proof of F in* PA, whereas $t{:}F$ is interpreted as *t is a proof of F in* PA.

Definition 6. A *proof predicate* is a provably Δ_1-formula $Prf(x,y)$ such that for every arithmetical sentence φ

$$\mathsf{PA} \vdash \varphi \quad \Leftrightarrow \quad \text{for some } n \in \omega \quad Prf(n, \ulcorner\varphi\urcorner) \text{ holds}^2.$$

A *provability predicate* $Pr(y)$ associated to $Prf(x,y)$ is $\exists x Prf(x,y)$. A comprehensive exposition of provability predicates can be found in [11]. A proof predicate *Prf(x,y)* is called *normal* if the following conditions are met:

Finiteness of proofs: for every k the set $T(k) = \{l \mid Prf(k,l)\}$ is finite. The function from k to the code of $T(k)$ is computable.

Conjoinability of proofs: for any k and l there is a natural number n such that

$$T(k) \cup T(l) \subseteq T(n).$$

Proposition 3. ([3]) *For every normal proof predicate Prf there are computable functions $\mathtt{m}(x,y)$, $\mathtt{a}(x,y)$, $\mathtt{c}(x)$ such that for all arithmetical formulas φ, ψ and all natural numbers k, n the following formulas are valid:*

$$Prf(k, \ulcorner\varphi{\rightarrow}\psi\urcorner) \wedge Prf(n, \ulcorner\varphi\urcorner) \rightarrow Prf(\mathtt{m}(k,n), \ulcorner\psi\urcorner)$$

$$Prf(k, \ulcorner\varphi\urcorner) \rightarrow Prf(\mathtt{a}(k,n), \ulcorner\varphi\urcorner), \qquad Prf(n, \ulcorner\varphi\urcorner) \rightarrow Prf(\mathtt{a}(k,n), \ulcorner\varphi\urcorner)$$

$$Prf(k, \ulcorner\varphi\urcorner) \rightarrow Prf(\mathtt{c}(k), \ulcorner Prf(k, \ulcorner\varphi\urcorner)\urcorner).$$

[2] By $\ulcorner\varphi\urcorner$ we denote a goedel number of φ.

The natural arithmetical proof predicate *PROOF(x,y)*

> "*x is the code of a derivation containing a formula with the code y*".

with appropriate functions $\mathtt{m}(x,y)$, $\mathsf{a}(x,y)$ and $\mathsf{c}(x)$ is the principal example of a normal proof predicate.

Definition 7. A provability interpretation $*$ of the language of LPGL has the following parameters:

- a normal proof predicate *Prf* with the functions $\mathtt{m}(x,y)$, $\mathsf{a}(x,y)$, $\mathsf{c}(x)$ as above,
- a mapping from propositional letters to sentences of arithmetic,
- a mapping from proof variables and proof constants to natural numbers.

Definition 8 We define an arithmetical translation F^* of proof terms t and LPGL-formulas F under a given provability interpretation $*$ by induction. By the previous definition, $*$ is defined on proof variables and proof constants as well as on propositional letters. In addition, $*$ commute with boolean connectives,

$$(t\cdot s)^* = \mathtt{m}(t^*,s^*), \quad (t+s)^* = \mathsf{a}(t^*,s^*), \quad (!t)^* = \mathsf{c}(t^*),$$

$$(t{:}F)^* = Prf(t^*,\ulcorner F^*\urcorner), \quad (\Box F)^* = \exists x Prf(x,\ulcorner F^*\urcorner).$$

An interpretation $*$ converts a proof polynomial t into a natural number t^*, an LPGL-formula F into an arithmetical sentence F^*. Given a constant specification *CS*, an arithmetical interpretation $*$ is a CS *-interpretation* if all formulas from *CS* are true (equivalently, are provable in PA) under interpretation $*$.

Theorem 3. (Arithmetical soundness) *If* LPGL *proves* F *then* $\mathsf{PA}\vdash F^*$ *for any arithmetical interpretation* $*$ *respecting constant specifications made during a given derivation of* F.

Proof. A straightforward induction on the derivation in LPGL. All axioms of and rules of LP and GL have been checked in [3] and [30] respectively. It remains to check the soundness of axioms C1-C3, out of which C1 and C2 are trivial. Let us check C3, i.e. $t{:}\Box F\rightarrow F$. If $Prf(t^*,\ulcorner Pr(\ulcorner F^*\urcorner)\urcorner)$ is true, then $Pr(\ulcorner F^*\urcorner)$ is also true and $\mathsf{PA}\vdash F^*$, hence $\mathsf{PA}\vdash Prf(t^*,\ulcorner Pr(\ulcorner F^*\urcorner)\urcorner)\rightarrow F^*$, i.e. $\mathsf{PA}\vdash (t{:}\Box F\rightarrow F)^*$. If $Prf(t^*,\ulcorner Pr(\ulcorner F^*\urcorner)\urcorner)$ is false, then $\mathsf{PA}\vdash\neg Prf(t^*,\ulcorner Pr(\ulcorner F^*\urcorner)\urcorner)$, hence $\mathsf{PA}\vdash Prf(t^*,\ulcorner Pr(\ulcorner F^*\urcorner)\urcorner)\rightarrow F^*$, i.e. again $\mathsf{PA}\vdash(t{:}\Box F\rightarrow F)^*$. □

Theorem 4. (Arithmetical completeness) *If* LPGL_{CS} *does not prove* F *then there exists a CS-interpretation* $*$ *such that* $\mathsf{PA}\nvdash F^*$.

Proof. A more or less straightforward simplified version of the arithmetical completeness proof for LPP from [28] works here. Basically, one has just to ignore the functions $\Downarrow^{\Box}$ and $\Uparrow_{\Box}$ of LPP and take into consideration the new principles C1 and C3 which do not alter the picture. □

Corollary 2. LPGL *is a conservative extension of* B.

2.4 Relations between LPGL and LPP

The language of joint logic of proofs and provability LPP from [27, 28] is an extension of the language of LP by modality $\Box$ and two additional unary functional symbols $\Downarrow^{\Box}$ and $\Uparrow_{\Box}$. Axioms of logic LPP consist of axioms of LP, axioms of GL and additional principles

B1. $t{:}A \rightarrow \Box(t{:}A)$,
B2. $\neg(t{:}A) \rightarrow \Box\neg(t{:}A)$,
B3. $t{:}\Box A \rightarrow (\Downarrow^{\Box} t){:}A$,
B4. $t{:}A \rightarrow (\Uparrow_{\Box} t){:}\Box A$.

The rules of LPP are the ones of LPGL.

The arithmetical semantics for LPP is defined similar to the one for LPGL (above) with additional arithmetical operations of proofs corresponding to $\Downarrow^{\Box}$ and $\Uparrow_{\Box}$. Main results about LPP were established in [27, 28], namely LPP was shown to be decidable, sound and complete with respect to the arithmetical provability semantics for each constant specification. The principal difference between LPP and LPGL is the presence of operations $\Downarrow^{\Box}$ and $\Uparrow_{\Box}$ in the former. As we have just shown, the main desired properties of a joint logic of proofs and provability can be achieved without extending the original joint language of LP and GL: the system LPGL does not need operations $\Downarrow^{\Box}$ and $\Uparrow_{\Box}$, but still captures both LP and GL, is arithmetical complete and closed under internalization.

In a certain sence LPGL is the minimal arithmetically complete system containing axioms and rules of LP and GL and closed under internalization. Indeed, LPGL is formulated in a joint language of LP and GL without adding any new operations. Axioms of LPGL represent arithmetically provable principles, hence they should all be included into the system. For any given constant specification the corresponding set of arithmetical instances of formulas in the langauge of LPGL is closed under R1, R2 and R3, hence whose rules should be included into the minimal system as well. Finally, by internalization, there should be a ground proof polynomial specified as a proof of each axiom of the minimal system; in LPGL it is a proof constant. The only design discretion we had was choosing an equivalent axiom system and using different constant specifications with basically the same outcome modulo to replacing some constants by corresponding ground proof terms.

Technically speaking, neither of LPGL and LPP is a subsystem of the other since each of them has some axioms A which are not axioms of the other system, hence the corresponding theorems $c{:}A$ of one system are not derivable of the other one. Clearly, we don't want to distinguish systems which have mutually derivable sets of axioms and rules or differ only by ground proof terms assigned to axioms.

There is a fair way to eliminate these undesirable effects. Both systems LPGL and LPP uniformly reduce to their fragments $\mathsf{LPGL}_{\emptyset}$ and $\mathsf{LPP}_{\emptyset}$ without constant specifications (Lemma 3 and Lemma 2.1 from [28]). Comparing $\mathsf{LPGL}_{\emptyset}$ and $\mathsf{LPP}_{\emptyset}$ tolerates equivalent changes in the axiom system or substituting proof constants by ground proof terms. With respect to this test, LPGL is a fragment of LPP.

Lemma 8. $\mathsf{LPGL}_\emptyset$ *is a subsystem of* $\mathsf{LPP}_\emptyset$.

Proof. All the rules and almost all the axioms of $\mathsf{LPGL}_\emptyset$, but C1 and C3, are the rules and axioms of $\mathsf{LPP}_\emptyset$ as well. To prove C3 in $\mathsf{LPP}_\emptyset$ we just repeat the derivation from Lemma 1. Here is a proof of C1 in $\mathsf{LPP}_\emptyset$:

1. $t{:}A \to A$, axiom B3;
2. $\Box t{:}A \to \Box A$, by reasoning in GL;
3. $t{:}A \to \Box t{:}A$, by B4 (positive introspection);
4. $t{:}A \to \Box A$, by propositional logic, from 2 and 3. □

We conjecture that LPP can be also imbedded into LPGL preserving the logical structure of the former. Namely, given a derivation $\mathcal{D}$ in LPP one can replace all occurrences of operations $\Downarrow^{\Box}(\cdot)$ and $\Uparrow_{\Box}(\cdot)$ by some LPGL-proof polynomials $\mathtt{u}(\cdot)$ and $\mathtt{d}(\cdot)$ respectively such that the resulting sequence $\mathcal{D}'$ is derivable in LPGL

3 Basic logics of knowledge with justifications

As we have already mentioned in Intruduction, there is a well developed approach due to Boolos-Goldblatt-Kuznetsov-Muravitsky [7, 15, 18, 19] of emulating S4 in the logic of formal provability, via so called "strong provability" modality $\boxdot F = F \wedge \Box F$. We will apply this translation to derive from the arithmetical provability semantics the basic epistemic logic with justifications, LPS4.

Definition 9. *Justification polynomials* for LPS4 are the same as proof polynomials for LPGL, i.e. they are terms built from *justification variables* $x, y, z, \ldots$ and *constants* $a, b, c, \ldots$ by means of three operations, *application* "·" (binary), *union* "+" (binary), and *evidence checker* "!" (unary). Formulas of the language of LPS4 are the same as of LPGL, i.e. they are defined by the grammar

$$A = S \mid A_1 \to A_2 \mid A_1 \wedge A_2 \mid A_1 \vee A_2 \mid \neg A \mid \Box A \mid t{:}A.$$

The *basic logic of knowledge with justifications*, LPS4, is a fusion of systems LP and S4, with principles connecting justifications and knowledge operators adopted from the joint logic of proofs and formal provability LPGL.

Definition 10. The system LPS4 has the following axioms and rules

I. **Classical propositional logic**

A1-A10. (the standard set of axioms, e.g. from [17])
R1. *Modus Ponens*

II. **Basic Epistemic Logic** S4

E1. $\Box(F \to G) \to (\Box F \to \Box G)$

E2. $\Box F \to \Box\Box F$
E3. $\Box F \to F$
R2. $\vdash F \;\Rightarrow \vdash \Box F$ (*necessitation*)

III. **Logic of Proofs** LP

LP1. $s{:}(F \to G) \;\to\; (t{:}F \to (s{\cdot}t){:}G)$ (*application*)
LP2. $t{:}F \;\to !t{:}(t{:}F)$ (*explicit introspection*)
LP3. $s{:}F \to (s{+}t){:}F, \quad t{:}F \to (s{+}t){:}F$ (*union of justifications*)
LP4. $t{:}F \to F$ (*reflexivity of explicit knowledge*)
R4. $\vdash c{:}A$, where A is an axiom and c is a proof constant (*constant specification*)

IV. **Principles connecting explicit and implicit knowledge**

C1. $t{:}F \to \Box F$ (*explicit-implicit connection*)
C2. $\neg(t{:}F) \to \Box\neg(t{:}F)$ (*negative introspection*)

Obviously, LPS4 contains both LP and S4. The principle of *weak reflexivity* $t{:}\Box F \to F$ is derivable in $\mathsf{LPS4}^-$, *explicit reflexivity* $t{:}F \to F$ is redundant but we keep it listed for convenience. Moreover, LPS4 is closed under substitutions of polynomials for justification variables and formulas for sentence variables, and enjoys the deduction theorem. For a given *constant specification* $CS = \{c_1{:}A_1, \ldots, c_n{:}A_n\}$ (where each A_i is an axiom from I-IV and each c_i is a proof constant) we define $\mathsf{LPS4}^-_{CS}$ and $\mathsf{LPS4}^-_{\emptyset}$ as in Definition 4.

Proposition 4. *Lemmas 2, 4, and 5, Propositions 1, 2 hold for* $\mathsf{LPS4}^-$.

Corollary 3. $\mathsf{LPS4}^-$ *enjoys internalization and constructive necessitation.*

Lemma 9. $\mathsf{LPS4}^-_{CS}$ *proves* $F \quad\Leftrightarrow\quad \mathsf{LPS4}^-_{\emptyset}$ *proves* $\bigwedge \mathrm{CS} \to F$.

Proof. Similar to Lemma 3, by induction on a derivation of F in $\mathsf{LPS4}^-_{CS}$. The only nontrivial case is the rule of necessitation which is handles by the positive introspection principle. □

An alternative formulation of $\mathsf{LPS4}^-$ (modulo substituting ground polynomials for justification constants) can be given by replacing the group IV by the principles of positive and negative introspection, or by the principle of decidability of evidences $\Box t{:}F \vee \Box\neg t{:}F$. Indeed, here is a derivation of C1 from I+II+III+positive and negative introspection:

1. $t{:}F \to F$, LP4
2. $\Box t{:}F \to \Box F$, by necessitation and E1
3. $t{:}F \to \Box t{:}F$, positive introspection
4. $t{:}F \to \Box F$, from 2 and 3.

Here is a derivation of positive and negative introspection from I+II+III+decidability of evidences:

1. $\Diamond t{:}F \to \Box t{:}F$, from decidability of evidences

2. $\Diamond\neg(t{:}F)\rightarrow\Box\neg(t{:}F)$, likewise
3. $t{:}F\rightarrow\Diamond t{:}F$, from E3
4. $\neg(t{:}F)\rightarrow\Diamond\neg(t{:}F)$, from E3
5. $t{:}F\rightarrow\Box t{:}F$, from 1,3
6. $\neg(t{:}F)\rightarrow\Box\neg(t{:}F)$, from 2,4.

The summary of this observation is

$$\mathsf{LPS4} = \mathsf{LP} + \mathsf{S4} + \textit{Positive and negative introspection}$$

$$= \mathsf{LP} + \mathsf{S4} + \textit{Decidability of explicit knowledge},$$

where the "equality" of logics is understood modulo to replacing some proof constants by ground proof polynomials.

3.1 Kripke models for $\mathsf{LPS4}^-$.

The logics of proofs and provability B [1], BGrz [23, 24], LPP [27, 28], LPGL (section 2) gave us a clear idea how to build Kripke models for $\mathsf{LPS4}^-$. In these cases a model is a Kripke model for the host modal logic (namely, GL) with formulas $t{:}F$ treated as additional atoms satisfying conditions of *stability* (every formula $t{:}F$ either holds at all nodes or does not hold at all nodes), *explicit reflexivity* ($t{:}F$ yields F) and some sort of reflexivity at the root node. In the case of S4-models we end up with the same picture, except for the root reflexivity which holds in an S4-model automatically.

Definition 11. An $\mathsf{LPS4}^-$-*model* is a triple $(K,\prec,\Vdash)$ where $(K,\prec)$ is an S4-frame (nonempty transitive and reflexive frame), $\Vdash$ is a forcing relation between nodes of K and $\mathsf{LPS4}^-$-formulas satisfying the *forcing conditions*.

1. usual modal conditions for $\Box$, i.e. $\Vdash$ respects boolean connectives at each node, $a\Vdash\Box F$ iff $b\Vdash F$ for all $b\succ a$;
2. *stability of explicit knowledge* every formula $t{:}F$ either holds at all nodes of K or does not hold at all nodes of K;
3. *reflexivity of explicit knowledge*: $a\Vdash t{:}F$ yields $a\Vdash F$;
4. if $a\Vdash s{:}(F\rightarrow G)$ and $a\Vdash t{:}F$ then $a\Vdash(s\cdot t){:}G$;
5. if $a\Vdash t{:}F$ then $a\Vdash !t{:}(t{:}F)$;
6. if $a\Vdash s{:}F$ then a forces both $(s+t){:}F$ and $(t+s){:}F$.

Item 1 shows that the epistemic modality behaves in $\mathsf{LPS4}^-$ in the normal way. Items 2 and 3 postulate decidability and reflexivity of explicit knowledge. Items 4-5 give the usual closure conditions on justifications.

A formula F holds in a model M if F holds at each node of M. M is an CS-model, for a given constant specification $CS=\{c_1{:}A_1,c_2{:}A_2\ldots c_n{:}A_n\}$ if all $c_i{:}A_i\in CS$ hold in M. A formula F is *CS-valid* if it holds in each CS-model.

From this definition it follows that whereas $a \Vdash \Box F$ is understood in the conventional manner as F *holds in all worlds accessible from a given* a, an assertion $a \Vdash t{:}F$ refers to all worlds in the model.

Theorem 5. (Soundness) *If A is derivable in* $\mathsf{LPS4}^-$*, then A is CS-valid for a constant specification CS associated with a given derivation of A.*

Proof. Immediate from the definitions. $\Box$

Theorem 6. (Completeness) *If A is not derivable in* $\mathsf{LPS4}^-_{CS}$*, then A is not CS-valid.*

Proof. We will use methods of building Kripke style models for logics of explicit proofs developed in [1],[2] and [28]. Without loss of generality we establish the completeness theorem for $\mathsf{LPS4}^-_{\emptyset}$, since, by Lemma 9, an $\mathsf{LPS4}^-_{\emptyset}$-countermodel for $\bigwedge CS \rightarrow F$ is at the same time a $\mathsf{LPS4}^-_{CS}$-countermodel for F. We restrict our considerations to the finite signature of symbols occurring in a given A, i.e. to a fixed finite set of sentence letters, justification variables and constants. So, the sets of polynomials and formulas of given length are finite. Let $|t|$ denote the length (the number of symbols) of t. For a finite set X of formulas by $|X|$ we understand the largest length $|t|$ of a term t such that $t{:}F \in X$. By $\mathtt{core}(X)$ we mean the set of all formulas of X of sort $t{:}F$.

Definition 12. A set X of $\mathsf{LPS4}^-$-formulas is *adequate* if X is closed under subformulas and

1. if $(s\cdot t){:}G$, $F \rightarrow G$ are in X, then $s{:}(F \rightarrow G)$, $t{:}F$ are also in X
2. if $(s+t){:}F$ is in X, then both $s{:}F$ and $t{:}F$ are in X.

It is easy to see that any finite set of formulas can be extended to a finite adequate set.

Definition 13. For a set X $\mathsf{LPS4}^-$-formulas by $\mathtt{compl}(X)$ we understand the minimal set containing X and such that

1. if $s{:}(F \rightarrow G)$, $t{:}F$ are in X then $(s\cdot t){:}G$ is in $\mathtt{compl}(X)$
2. if $t{:}F$ is in X then $!t{:}t{:}F$ is in $\mathtt{compl}(X)$
3. if $s{:}F$ is in X and $|t| \le |X|$, then both $(s+t){:}F$ and $(t+s){:}F$ are in $\mathtt{compl}(X)$.

Let $X_\infty = X_0 \cup X_1 \cup X_2 \cup \ldots$ where $X_0 = X$ and $X_{i+1} = \mathtt{compl}(X_i)$ for all $i = 0, 1, 2, \ldots$. Such X_∞ is called the *completion* of X. An $\mathsf{LPS4}^-$ model $(K, \prec, \Vdash)$ is finitely generated if

$$K \Vdash t{:}F \quad \Leftrightarrow \quad t{:}F \in \mathtt{core}(X_\infty)$$

for some finite set X.

We will establish the completeness of $\mathsf{LPS4}^-_{CS}$ with respect to finitely generated models, which will yield a decidability of $\mathsf{LPS4}^-_{CS}$.

Lemma 10. *Given a finite adequate set X and a structure $\mathcal{K} = (K, \prec, \Vdash)$ such that all conditions from Definition 11 restricted to X hold for $\mathcal{K}$ there is a finitely generated* $\mathsf{LPS4}^-$*-model $\mathcal{K}_1 = (K, \prec, \Vdash_1)$ which coincides with $\mathcal{K}$ on formulas from X, i.e. for any $a \in K$ and $F \in X$*

$$a \Vdash F_1 \quad \Leftrightarrow \quad a \Vdash F.$$

Proof. Consider $Y = \{t{:}F \in X \mid K \Vdash t{:}F\}$ and take its completion Y_∞. Define $\Vdash_1$ as

$$a \Vdash_1 S \quad \Leftrightarrow \quad a \Vdash S$$

for any sentence variable S and

$$a \Vdash_1 t{:}F \quad \Leftrightarrow \quad t{:}F \in Y_\infty.$$

Naturally, $\Vdash_1$ is extended to all formulas by the usual modal truth tables from Definition 11(1) of an $\mathsf{LPS4}^-$-model. First of all, we notice that $t{:}F \in Y_\infty$ yields $F \in X$. We also observe that

$$a \Vdash_1 G \quad \Leftrightarrow \quad a \Vdash G \tag{4}$$

for all formulas G from X. Indeed, this can be shown by induction on G. Base case 1, G is a sentence letter, is covered by the definition. Base case 2: G is $t{:}F$. Obviously, $a \Vdash t{:}F$ yields $a \Vdash_1 t{:}F$. Suppose $t{:}F \in X$, $a \Vdash_1 t{:}F$ and $a \nVdash t{:}F$. In this situation $t{:}F$ has appeared in Y_∞ by the completion process. There are three possibilities: $t{:}F$ is $(u \cdot v){:}F$, $(u+v){:}F$ and $!s{:}s{:}H$. If $t{:}F$ is $(u \cdot v){:}F$, then $u{:}(B \rightarrow F) \in Y_\infty$ and $v{:}B \in Y_\infty$ for some B. By the previous observation, $(B \rightarrow F) \in X$, hence, since X is adequate, $u{:}(B \rightarrow F)$ and $v{:}B$ are both in X. By the IH, $a \Vdash u{:}(B \rightarrow F)$ and $a \Vdash v{:}B$. Since condition 4 of Definition 11 is met for $\Vdash$ on X, $a \Vdash (u{\cdot}v){:}F$, hence $a \Vdash t{:}F$. The remaining clauses $(u{+}v){:}F$ and $!s{:}s{:}H$ are treated in the same manner. Inductive steps employ the same conditions for both $\Vdash$ and $\Vdash_1$. This concludes the proof of observation (4).

Now we claim that $\mathcal{K}_1 = (K, \prec, \Vdash_1)$ is a desired model for lemma 10. We first check that $\mathcal{K}_1$ is a model. Condition 1 is met by the definition of $\Vdash_1$. Condition 2 (stability) is guaranteed by the fact that $a \Vdash_1 t{:}F$ does not depend on a.

Let us check Condition 3 (explicit reflexivity). Suppose this condition is violated and consider the first moment in the building of Y_∞ when $a \Vdash_1 t{:}F$ and $a \nVdash_1 F$ for some a and $t{:}F$. Case 1: $t{:}F \in Y$, i.e. $a \Vdash t{:}F$. Then $t{:}F \in X$, and, since X is an adequate set, $F \in X$. By lemma assumptions, $\Vdash$ is reflexive on formulas from X, hence $a \Vdash F$ and $a \Vdash_1 F$. Case 2: $t{:}F$ is $(u{\cdot}v){:}F$ and $u{:}(G \rightarrow F)$ and $v{:}G$ have appeared in Y_∞ earlier. By the definition of $\mathcal{K}_1$, $a \Vdash_1 u{:}(G \rightarrow F)$ and $a \Vdash_1 v{:}G$. By the IH, $a \Vdash_1 G \rightarrow F$ and $a \Vdash_1 G$, hence $a \Vdash_1 F$. Case 3: $t{:}F$ is $(u{+}v){:}F$ where one of the formulas $u{:}F$ or $v{:}F$ has appeared in Y_∞ earlier and $a \Vdash_1 u{:}F$ or $a \Vdash_1 v{:}F$. By the IH, $a \Vdash_1 F$. Case 4: $t{:}F$ is $!s{:}s{:}G$ and $s{:}G$ has appeared in Y_∞ earlier. By IH, $a \Vdash_1 s{:}G$, hence $a \Vdash_1 F$.

Conditions 4-6 of Definition 11 are met because Y_∞ is closed under operational axioms LP1-LP3.

Since Y and Y_∞ consist of formulas $t{:}F$ only, $\mathtt{core}(Y_\infty) = Y_\infty$ and model $\mathcal{K}_1$ is finitely generated by Y. □

Lemma 11. *If A is not derivable in $\mathsf{LPS4}^-_{CS}$, then there is a finite adequate set X containing A and the structure $\mathcal{K} = (K, \prec, \Vdash)$ such that all conditions from Definition 11 restricted to X hold for $\mathcal{K}$ and A is not valid in $\mathcal{K}$.*

Proof. Let X be the smallest adequate set containing A. Apparently, X is finite. Consider fresh sentence variables $S_{t:F}$ for each formula $t{:}F \in X$. Let σ be a substitution $[S_{t:F}/t{:}F]$ for all $t{:}F \in X$. To every $B \in X$ we associate an $\mathsf{S4}$-formula B^t such that $B = B^t\sigma$, i.e. B is the result of substituting all $t{:}F$'s for $S_{t:F}$'s in B^t. Now we encode the adequacy conditions on X by a propositional $\mathsf{S4}$-formula. Let Y be the conjunction of the following $\mathsf{S4}$-formulas:

1. $S_{t:F} \rightarrow F^t$
2. $S_{s:(F\rightarrow G)} \rightarrow (S_{t:F} \rightarrow S_{(s\cdot t):G})$
3. $S_{t:F} \rightarrow S_{!t:t:F}$
4. $S_{t:F} \rightarrow S_{(s+t):F}$, $S_{t:F} \rightarrow S_{(t+s):F}$
5. $S_{t:F} \rightarrow \Box F^t$
6. $\neg S_{t:F} \rightarrow \Box\neg S_{t:F}$

The substitution σ makes Y some set of axioms of $\mathsf{LPS4}^-_{\emptyset}$. Moreover, $\mathsf{S4} \nvdash \bigwedge Y \rightarrow A^t$, since otherwise an $\mathsf{S4}$-proof of $\bigwedge Y \rightarrow A^t$ under substitution σ would became a proof of A in $\mathsf{LPS4}^-_{\emptyset}$. Apply the Kripke completeness theorem for $\mathsf{S4}$ and find a finite $\mathsf{S4}$-contermodel $\mathcal{K}_0 = (K, \prec, \Vdash_0)$ for $\bigwedge Y \rightarrow A^t$. By definition, all formulas from Y hold at each node of K. Now keep the frame $(K, \prec)$ and define $\mathcal{K} = (K, \prec, \Vdash)$ by stipulating

$$\Vdash S \quad \Leftrightarrow \quad \Vdash_0 S \qquad \text{and} \qquad \Vdash t : F \quad \Leftrightarrow \quad \Vdash_0 S_{t:F} \tag{5}$$

Obviously, for all $F \in X$

$$\Vdash F \quad \Leftrightarrow \quad \Vdash_0 F^t,$$

hence the resulting $\mathcal{K}$ is a desired model: all forcing conditions from Definition 11 are met on X, A does not hold in $\mathcal{K}$. □
Theorem 6 follows now immediately from lemma 10 and lemma 11. □

Corollary 4. $\mathsf{LPS4}^-_{CS}$ *is decidable for each constant specification CS.*

Proof. By Post argument, since countermodels in $\mathsf{LPS4}^-_{CS}$ are finitely generated, hence efficiently described. □

3.2 Arithmetical semantics for $\mathsf{LPS4}^-$.

Arithmetical semantics for $\mathsf{LPS4}^-$ is provided by the strong provability operator defined in the provability logic as $\boxdot F := F \wedge \Box F$. Consider the arithmetical provability semantics for LPGL in Peano Arithmetic PA (subsection 7) where $\Box F$ is interpreted as *there is a proof of F in* PA and $t{:}F$ as *t is a proof of F in* PA. We recast this to a provability semantics for $\mathsf{LPS4}^-$ where $\Box F$ is interpreted as

F is true and provable in PA,

whereas $t{:}F$ is interpreted as before:

t is a proof of F in PA.

Definition 14. Define a translation $()^+$ of $\mathsf{LPS4}^-$ formulas into the language of LPGL

$$S^+ = S,\ \ (A \to B)^+ = (A \to B),\ \ (\neg A)^+ = \neg A,\ \ (t{:}A)^+ = t{:}A,\ \ (\Box A)^+ = A \wedge \Box A.$$

An arithmetical provability semantics of $\mathsf{LPS4}^-$ is inherited from the one of the logic of proofs and formal provability LPGL: in Definition 7 the item corresponding to the modality should be altered to the strong provability reading:

$$(\Box F)^* = F^* \wedge \exists x Prf(x, \ulcorner F^* \urcorner).$$

Lemma 12. *If* $\mathsf{LPS4}^- \vdash F$ *then* $\mathsf{LPGL} \vdash F^+$.

Proof. Induction of derivations in $\mathsf{LPS4}^-$. Translations of the axioms E1-E3 of $\mathsf{S4}$ are all derivable in GL [7]. Axioms of LP remain axioms of the same sort after the translation. Let us check the connection axioms of $\mathsf{LPS4}^-$: both of them

$$(t{:}F \to \Box F)^+ = t{:}F \to F \wedge \Box F$$

and

$$[\neg(t{:}F) \to \Box\neg(t{:}F)]^+ = \neg(t{:}F) \to (\neg(t{:}F) \wedge \Box\neg(t{:}F))$$

are obviously derivable in $\mathsf{LPGL}_\emptyset$, hence in LPGL. The rules are straightforward. □

Theorem 7. (Arithmetical soundness of logic of knowledge with justifications) *If* $\mathsf{LPS4}^-$ *proves F then* $\mathsf{PA} \vdash F^*$ *for any arithmetical interpretation* $*$.

Proof. A combination of theorem 3 and lemma 12. □

3.3 A complete system of strong provability with justifications.

An arithmetically complete system $\mathsf{LPS4Grz}^-$ of the strong provability with proofs can be axiomatized by adding to $\mathsf{LPS4}^-$ the modal axiom by Grzegorczyk $\Box(\Box(F \to \Box F) \to F) \to F$. Kripke models for $\mathsf{LPS4Grz}^-$ are the special sort of $\mathsf{LPS4}^-$-models when the frame is a reflexive partial order. These facts can be established by a straightforward combination of methods from [23, 24] and the current paper.

3.4 Minimal epistemic logic with justifications LPS4.

Consider the fragment of $\mathsf{LPS4}^-$, which we denote $\mathsf{LPS4}$, obtained by omitting axiom C2 from the Definition 10 of $\mathsf{LPS4}^-$ above. In other words,

$$\mathsf{LPS4}^- = \quad \mathsf{LP} + \mathsf{S4} + (t{:}F \to \Box F) \quad =$$

$$= \quad \mathsf{LP} + \mathsf{S4} + \textit{positive introspection.}$$

The logic $\mathsf{LPS4}$ is perhaps the minimal epistemic logic with justifications when no specific assumptions were made concerning the character of the explicit knowledge operators.

As a formal system $\mathsf{LPS4}$ behaves normally: it is closed under substitutions, enjoys the deduction theorem, internalization, has reasonable Kripke models similar to the ones for $\mathsf{LPS4}^-$ described above, is sound with respect to the arithmetical semantics of the strong provability similar to the one for $\mathsf{LPS4}^-$. Furthermore, $\mathsf{LPS4}$ has some interesting features not present in $\mathsf{LPS4}^-$.

Theorem 8. $\mathsf{LPS4}$ *enjoys Fitting semantics.*

Proof. Fitting semantics from [12] was designed to model LP_{CS} for a fixed constant specification CS; this observation secures the "LP-part" of $\mathsf{LPS4}$. Furthermore, Fitting models have S4-frames, which guarantees the soundness of the "S4-part" of $\mathsf{LPS4}$. The principle $t{:}F \to \Box F$ holds at each mode of a Fitting model by the definition of Fitting's forcing: $a \Vdash t{:}F$ iff t is an acceptable witness for F at a and $b \Vdash F$ for all nodes b accessible from a. The second of those conditions immediately yields that $a \Vdash \Box F$ as well. $\Box$

We conclude with formulating some open problems.

Problem 1. Whether $\mathsf{LPS4}$ is compete with respect to the Fitting semantics?

Problem 2. Whether $\mathsf{LPS4}$ enjoys the *realization* property: given a derivation D in $\mathsf{LPS4}$ with a given constant specification CS one could find a realization r of all occurrences of $\Box$ in D and a new constant specification CS' containing CS such that the resulting formula F^r is derivable in $\mathsf{LP}_{CS'}$?

Problem 3. Find topological semantics for LPGL, $\mathsf{LPS4}$ and $\mathsf{LPS4}^-$ that extends Tarski topological semantics for S4 (cf. [20]) and Esakia topological semantics for GL (cf. [9]).

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