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THE TOPOLOGY OF JUSTIFICATION

Abstract. Justification Logic is a family of epistemic logical systems obtained from modal logics of knowledge by adding a new type of formula $t:F$, which is read *t is a justification for F*. The principal epistemic modal logic **S4** includes Tarski’s well-known topological interpretation, according to which the modality $\Box X$ is read *the Interior of X in a topological space* (the topological equivalent of the ‘*knowable part of X*’). In this paper, we extend Tarski’s topological interpretation from **S4** to Justification Logic systems with both modality and justification assertions. The topological semantics interprets $t:X$ as a reachable subset of X (the topological equivalent of ‘*test t confirms X*’). We establish a number of soundness and completeness results with respect to Kripke topology and the real topology for **S4**-based systems of Justification Logic.

Keywords: Modal logic, Justification Logic, topological semantics, Tarski.

1. Introduction

The Justification Logic is a family of logical systems originated from the Logic of Proofs LP (cf. [2, 3, 5, 9, 10]). These systems are normally obtained from epistemic modal logics by adding new type of formula $t:F$ which is read as

t is a justification for F.

The standard provability semantics for LP was given in [2, 3] and it follows Gödel’s design from [19].

The epistemic Kripke-style semantics for LP was found in [16, 17] and later extended to Justification Logic systems containing both epistemic modalities

for ‘ F is known’ and justification assertions ‘ t is a justification for F ’ ([9, 10]). The major epistemic modal logic **S4** has a well-known Tarski’s topological interpretation. Such a connection between modal logic and topology proved to be fruitful for both domains. In particular, topology was used in [25] to prove Gödel’s conjecture about Gödel’s embedding of Intuitionistic Logic to **S4**. On the other hand, **S4**-based systems of modal logic have been used to describe the behavior of dynamic systems in real topology (cf. [14]).

Fundamental connections between modal logic to topology date back to Kuratowski [21] and Riesz in [32]. Let

$$\mathcal{T} = \langle \mathbf{X}, \mathbb{I} \rangle$$

be a topological space, where \mathbf{X} is a set and \mathbb{I} the interior operation. The following principles hold for all subsets Y and Z of \mathbf{X} :

1. $\mathbb{I}(Y \cap Z) = \mathbb{I}Y \cap \mathbb{I}Z$,
2. $\mathbb{I}Y = \mathbb{I}\mathbb{I}Y$,
3. $\mathbb{I}Y \subseteq Y$,
4. $\mathbb{I}\mathbf{X} = \mathbf{X}$.

These principles can be presented as propositional modal formulas: set-theoretical operations are represented by the corresponding Boolean connectives, and the interior operator \mathbb{I} by the modality \Box :

1. $\Box(A \wedge B) = \Box A \wedge \Box B$,
2. $\Box A \rightarrow \Box \Box A$,
3. $\Box A \rightarrow A$,
4. $\Box \top$.

These are the postulates of the modal logic **S4**. This correlation was noticed in the late 1930s by Tarski, Stone, and Tang. Neither Lewis’ original approach to modal logic ([22, 23]), nor Gödel’s provability interpretation of **S4** ([18]) were related to topology.

Tarski’s topological interpretation of propositional modal logic naturally extends the set-theoretical interpretation of classical propositional logic. Given a topological space $\mathcal{T} = \langle \mathbf{X}, \mathbb{I} \rangle$ and a valuation (mapping) $*$ of propositional letters to subsets of \mathbf{X} , we can extend $*$ to all modal formulas as

follows:

$$\begin{aligned}
 (\neg A)^* &= \mathbf{X} \setminus A^*, \\
 (A \wedge B)^* &= A^* \cap B^*, \\
 (A \vee B)^* &= A^* \cup B^*, \\
 (\Box A)^* &= \mathbb{I}A^*.
 \end{aligned} \tag{1}$$

A formula A is called valid in \mathcal{T} (notation: $\mathcal{T} \Vdash A$) if $A^* = \mathbf{X}$ for any valuation $*$. The set

$$\mathbf{L}(\mathcal{T}) := \{A \mid \mathcal{T} \Vdash A\}$$

is called the modal logic of \mathcal{T} . The following classical result in this area is due to McKinsey and Tarski ([24]):

THEOREM. *If S is a separable dense-in-itself metric space, then $\mathbf{L}(S) = \mathbf{S4}$.*

In particular, for each $n = 1, 2, 3, \dots$,

$$\mathbf{L}(\mathbb{R}^n) = \mathbf{S4}.$$

Simplified proofs of this theorem were obtained independently in [12, 26, 34]. Kripke models for modal logics can be regarded a special case of topological models. Indeed, given a Kripke frame $\langle W, R \rangle$, one can construct a topological space $\langle W, \mathbb{I} \rangle$ where

$$\mathbb{I}U := \{x \mid R(x) \subseteq U\},$$

so that validities in these two models are the same. Hence, Kripke-completeness yields the topological completeness.

The Justification Logic grew from the Logic of Proofs LP. The first incomplete sketch of the Logic of Proofs was made in Gödel's lecture of 1938 [19], which remained unpublished until 1995 when the full Logic of Proofs was rediscovered independently in [2]. The Logic of Proofs LP ([2, 3, 4, 6, 15]) introduces the notion of *proof polynomials*, i.e., terms built from proof variables and constants by means of three operations:

- *application* “ \cdot ”, which given a proof s of an implication $F \rightarrow G$ and a proof t of its antecedent F provides a proof $s \cdot t$ of the succedent G ;
- *sum* “ $+$ ”, which given proofs s and t returns a proof $s + t$ of everything proven by s or t ;
- *proof checker* “ $!$ ”, which given a proof t of F verifies it and provides a proof $!t$ of the fact that t is indeed a proof of F .

LP is a system with classical logic enriched by additional atoms

$$t:F$$

where t is a proof polynomial and F is a formula, with the intended reading

$$t \text{ is a proof of } F.$$

As it was shown in [2, 3], LP describes all valid principles of proof operators $t:F$

$$t \text{ is a proof of } F \text{ in Peano Arithmetic}$$

in its language.

The papers [1, 8, 28, 29, 30, 31, 33, 35] studied joint logics of proofs and provability in a format that includes both provability assertions $\Box F$ and proof assertions $t:F$.

In [5, 8, 9, 10] this approach has been extended to *epistemic logic* and applied for building mathematical models of justification, knowledge and belief. In the epistemic context, proof terms are called *justification terms*, the set of possible operations on justifications is no longer limited to $\{\cdot, +, !\}$, and sentences $t:F$ are interpreted as

$$t \text{ is a justification for } F.$$

In particular, [9] introduced and studied the basic epistemic logic with justifications,

$$\text{S4LP} = \text{S4} + \text{LP} + (t:F \rightarrow \Box F).$$

Epistemic models for Justification Logics has been developed in [5, 8, 9, 10, 16, 17, 27]. A *Fitting model* for S4LP is $\langle W, R, \mathcal{A}, \Vdash \rangle$, where

- $\langle W, R \rangle$ is an S4-frame;
- \mathcal{A} is an admissible evidence function: for each term t and formula F , $\mathcal{A}(t, F)$ is a subset of W . Informally, $\mathcal{A}(t, F)$ specifies a set of worlds where t is an admissible evidence for F . An evidence function is assumed to be monotonic:

$$u \in \mathcal{A}(t, F) \text{ and } uRv \text{ yield } v \in \mathcal{A}(t, F)$$

and has natural closure properties that agree with operations of S4LP;

- \Vdash behaves in the standard Kripke style on Boolean connectives and \Box :
 - $u \Vdash p$ or $u \not\Vdash p$ is specified for each world u and each propositional variable p ;
 - $u \Vdash F \wedge G$ iff $u \Vdash F$ and $u \Vdash G$, $u \Vdash F \vee G$ iff $u \Vdash F$ or $u \Vdash G$, $u \Vdash \neg F$ iff $u \not\Vdash F$;
 - $u \Vdash \Box F$ iff $v \Vdash F$ for all v such that uRv ;
- $u \Vdash t:F$ iff $u \Vdash \Box F$ and $u \in \mathcal{A}(t, F)$.

In [8, 17], S4LP is shown to be sound and complete with respect to this epistemic semantics.

2. Topological Semantics for Justifications

We start with offering a topological semantics for operation-free single modality Justification Logics. It means we will work with the usual language of propositional modal logic enriched by a new construction $t:F$ where t is a justification variable and F is a formula.

An interpretation is defined for a topological space $\mathcal{T} = \langle \mathbf{X}, \mathbb{I} \rangle$ supplied with a *test function* \mathcal{M} which maps a term t and a formula F to

$$\mathcal{M}(t, F) \subseteq \mathbf{X}.$$

The informal meaning of \mathcal{M} is that $\mathcal{M}(t, F)$ represents a “potentially accessible” region of \mathbf{X} associated with F and t .

We assume that an evaluation $*$ maps propositional variables to subsets of \mathbf{X} , $*$ works on Boolean connectives and modality \Box according to aforementioned Tarski’s interpretation (1). We will study several natural ways to extend $*$ on formulas $t:F$ and corresponding subsystems of S4LP. This approach was first discussed in [11].

We build our topological semantics for the Justification Logic language on the following formal and informal assumptions.

1. Our semantics is based on Tarski’s topological semantics (1), e.g.,

$$(\Box F)^* = \mathbb{I}(F^*).$$

2. Justification terms are symbolic representations of *tests*. We postulate existence of a test function \mathcal{M} which for each t and F specifies a set of points $\mathcal{M}(t, F)$ which we call

the set of possible outcomes of a test t of a property F .

3. The $t:F$ will return a set of points where a test t confirms F . This reading will be supported by definitions (for different subsystems of S4LP):

$$(t:F)^* = F^* \cap \mathcal{M}(t, F) \quad (2)$$

or

$$(t:F)^* = \mathbb{I}(F^*) \cap \mathcal{M}(t, F). \quad (3)$$

In case (2), test t supports F at all points where the possible outcome of t lies inside F . Case (3) corresponds to the “robust” understanding of testing: test t supports F at all points of the possible outcome of t which lie in the interior of F .

4. We first consider systems without operations on tests.

Now we introduce several systems of Justification Logic and simultaneously define their topological semantics in format $\langle \mathcal{T}, \mathcal{M} \rangle$, where $\mathcal{T} = \langle \mathbf{X}, \mathbb{I} \rangle$ is a topological space and \mathcal{M} is a test function on \mathbf{X} .

2.1. Basic Testing System S4B₀

The most basic system in our list is

$$\text{S4B}_0 = \text{S4} + (t:F \rightarrow F).$$

In this system, there are no any assumptions about tests; they don't necessarily produce open sets of outcomes. The only requirement on tests is that the set of values where a test t confirms F is consistent with F itself. A topological model for S4B₀ is a topological space $\mathcal{T} = \langle \mathbf{X}, \mathbb{I} \rangle$ with an arbitrary test function \mathcal{M} on \mathbf{X} , and an evaluation $*$ of propositional letters as subsets of \mathbf{X} . Boolean connectives are interpreted in the usual set-theoretical way, $(\Box F)^* = \mathbb{I}(F^*)$. The definition of the test assertion is as follows:

$$(t:F)^* = F^* \cap \mathcal{M}(t, F).$$

An S4B₀-formula is true in a model if it is evaluated as the whole space \mathbf{X} . A formula is valid if it is true in every model.

2.2. Robust Testing System S4B₁

The next system under consideration is

$$\text{S4B}_1 = \text{S4} + (t:F \rightarrow \Box F).$$

In $S4B_1$ -models, test sets are still arbitrary (not necessarily open); however, the test assertions are interpreted as “robust inclusion” (3), i.e. the set of values where test t confirms F is a subset of the interior of F :

$$(t:F)^* = \mathbb{I}(F^*) \cap \mathcal{M}(t, F).$$

2.3. Robust Open Testing System $S4B_2$

Finally, we consider

$$S4B_2 = S4 + (t:F \rightarrow F) + (t:F \rightarrow \Box t:F).$$

This system corresponds to the full operation-free version of $S4LP$. In $S4B_2$ -models, test sets are open, test assertions are interpreted in the robust sense (3):

$$(t:F)^* = \mathbb{I}(F^*) \cap \mathcal{M}(t, F).$$

2.4. Topological Soundness and Completeness

THEOREM 1. *All three systems $S4B_0$, $S4B_1$, and $S4B_2$ are sound and complete with respect to the corresponding classes of topological models.*

PROOF. *Soundness of $S4B_0$.* Since the propositional modal part of $S4B_0$ has been copied from $S4$, the soundness of the $S4$ -axioms and rules is checked similarly to those in Tarski’s topological interpretation of $S4$. It remains to check that the only additional axiom of $S4B_0$, $t:F \rightarrow F$, is $S4B_0$ -valid. For this, it suffices to check that $(t:F)^* \subseteq F^*$, for each evaluation $*$, which is immediate from the definition of $(t:F)^* = F^* \cap \mathcal{M}(t, F)$.

Soundness of $S4B_1$. The same as for $S4B_0$, except that now we have to check the validity of $t:F \rightarrow \Box F$. In $S4B_1$ -models, $(t:F)^* = \mathbb{I}(F^*) \cap \mathcal{M}(t, F)$ and $(\Box F)^* = \mathbb{I}(F^*)$, hence $(t:F)^* \subseteq (\Box F)^*$ and $(t:F \rightarrow \Box F)^* = \mathbf{X}$.

Soundness of $S4B_2$. The same as before, but now we have to check the validity of both $t:F \rightarrow F$ and $t:F \rightarrow \Box t:F$. The former follows from the definition of $(t:F)^*$ as $\mathbb{I}(F^*) \cap \mathcal{M}(t, F)$. To establish the latter, note that in $S4B_2$ -models, test sets are open, hence each $(t:F)^*$ is open, as the intersection of two open sets $\mathbb{I}(F^*)$ and $\mathcal{M}(t, F)$. Therefore, $(t:F)^* = \mathbb{I}(t:F)^*$ and $(t:F \rightarrow \Box t:F)^* = \mathbf{X}$.

Completeness proof goes via epistemic models which are then converted into topological spaces with the cone topology.

We consider the case of $S4B_2$, the remaining cases are receiving a similar or more simple treatment. Let us first establish the completeness of

$S4B_2$ with respect to Fitting models $\langle W, R, \mathcal{A}, \Vdash \rangle$ such that $\langle W, R \rangle$ is an $S4$ -frame, \mathcal{A} is a monotonic admissible evidence function: for each term t and formula F ,

$$u \in \mathcal{A}(t, F) \text{ and } uRv \text{ yield } v \in \mathcal{A}(t, F).$$

In such models, \Vdash behaves in the standard Kripke style on Boolean connectives and \Box , and

$$u \Vdash t:F \quad \text{iff} \quad u \Vdash \Box F \text{ and } u \in \mathcal{A}(t, F).$$

Now we describe the canonical Fitting model for $S4B_2$ by the maximal consistent set construction.

- W is the set of all maximal consistent sets in $S4B_2$. We denote elements of W as Γ, Δ , etc.;
- $\Gamma R \Delta$ iff $\Gamma^\sharp \subseteq \Delta$, where $\Gamma^\sharp = \{\Box F \mid \Box F \in \Gamma\}$;
- $\mathcal{A}(s, F) = \{\Gamma \in W \mid s:F \in \Gamma\}$;
- $\Gamma \Vdash p$ iff $p \in \Gamma$.

Let us check that $\langle W, R, \mathcal{A}, \Vdash \rangle$ is indeed an $S4B_2$ -model. It is immediate from the definitions that the accessibility relation R is reflexive and transitive. In addition, the admissible evidence function \mathcal{A} is monotonic. Indeed, suppose $\Gamma \in \mathcal{A}(t, F)$ and $\Gamma R \Delta$. Then $t:F \in \Gamma$, $\Box t:F \in \Gamma$, $\Box t:F \in \Delta$, and $t:F \in \Delta$, i.e. $\Delta \in \mathcal{A}(t, F)$.

LEMMA 1 (Truth Lemma). *For every formula F , $\Gamma \Vdash F$ iff $F \in \Gamma$.*

Proof. Induction on F . The base case is given in the definition of the canonical model. The Boolean and modality cases are standard.

Let us consider the case when F is $t:G$. Let $t:G \in \Gamma$ and $\Gamma R \Delta$. Then $\Box t:G \in \Gamma$, $\Box t:G \in \Delta$, $t:G \in \Delta$ (since $\Box t:G \rightarrow t:G \in \Delta$), and $G \in \Delta$ (since $t:G \rightarrow G \in \Delta$). By the Induction Hypothesis, $\Delta \Vdash G$. Furthermore, by the definition of the admissible evidence function, $\Gamma \in \mathcal{A}(t, G)$, hence $\Gamma \Vdash t:G$.

If $t:G \notin \Gamma$, then $\Gamma \notin \mathcal{A}(t, G)$, hence $\Gamma \not\Vdash t:G$. ◻

Let us now finish the proof of completeness of $S4B_2$ with respect to $S4B_2$ -models. Suppose $S4B_2 \not\models F$. Then the set $\{\neg F\}$ is consistent, and hence included into some maximal consistent set Γ . Naturally, $F \notin \Gamma$. By the Truth Lemma, $\Gamma \not\Vdash F$.

Now we convert a given countermodel $\mathcal{K} = \langle W, R, \mathcal{A}, \Vdash \rangle$ for F into an appropriate topological space and find an interpretation under which F does

not hold. A Kripke topological space $\mathcal{T}_{\mathcal{K}}$ associated with \mathcal{K} is a topological space with the carrier W and open sets which are all subsets of W closed upward under R :

Y is open iff for all $u \in Y$, if uRv then $v \in Y$.

To make $\mathcal{T}_{\mathcal{K}}$ a topological $\mathbf{S4B}_2$ -model it remains to define a test function

$$\mathcal{M}(t, F) = \mathcal{A}(t, F).$$

Given a Fitting model $\mathcal{T}_{\mathcal{K}} = \langle W, R, \mathcal{A}, \Vdash \rangle$ for $\mathbf{S4B}_2$ we can also define a topological interpretation $*$ of $\mathbf{S4B}_2$ -language in $\mathcal{T}_{\mathcal{K}}$:

$$p^* = \{u \in W \mid u \Vdash p\} \text{ for a propositional letter } p.$$

Any interpretation $*$ is extended to all $\mathbf{S4B}_2$ -formulas in the standard way:

- $(A \vee B)^* = A^* \cup B^*$;
- $(\neg A)^* = W \setminus A^*$;
- $(\Box A)^* = \mathbb{I}(A^*)$;
- $(t : A)^* = \mathbb{I}(A^*) \cap \mathcal{M}(t, A)$.

From the definitions it is immediate that $t:G \rightarrow G$ holds in this model. Note that due to monotonicity of the admissible evidence function \mathcal{A} , for each t and F , the test sets $\mathcal{M}(t, G)$ are open in $\mathcal{T}_{\mathcal{K}}$. Therefore $t:G \rightarrow \Box t:G$ also holds at the model.

LEMMA 2 (The Main Lemma). *$u \Vdash G$ iff $u \in G^*$.*

Proof. Induction on G . The base case when G is atomic is covered by the definition. The Boolean connective case is straightforward.

Let G be $\Box B$. Suppose $u \Vdash \Box B$, then for all $v \in W$ such that uRv , $v \Vdash B$ as well. By the Induction Hypothesis, $v \in B^*$ for all $v \in W$ such that uRv . This yields that the whole open cone $O_u = \{v \mid uRv\}$ is a subset of B^* . Therefore, $u \in \mathbb{I}(B^*) = (\Box B)^*$.

Suppose $u \in (\Box B)^* = \mathbb{I}(B^*)$. Since $\mathbb{I}(B^*)$ is open, $v \in \mathbb{I}(B^*)$ hence $v \in B^*$, for all v such that uRv . By the Induction Hypothesis, $v \Vdash B$ for all v such that uRv . Therefore, $u \Vdash \Box B$.

Let G be $t:B$. Suppose $u \Vdash t:B$. Then, by definition, $u \in \mathcal{A}(t, B)$ and $v \Vdash B$, for all v such that uRv . By the definition of a test function, $u \in \mathcal{M}(t, B)$. By the Induction Hypothesis, $v \in B^*$ for all v such that uRv , which means that $u \in \mathbb{I}(B^*)$. Hence $u \in \mathbb{I}(B^*) \cap \mathcal{M}(t, B)$, i.e., $u \in (t:B)^*$.

Suppose $u \in \mathbb{I}(B^*) \cap \mathcal{M}(t, B)$. Then $u \in \mathcal{M}(t, B)$ hence $u \in \mathcal{A}(t, B)$. Furthermore, $u \in \mathbb{I}(B^*)$. Like in the case $G = \Box B$, we conclude that $u \Vdash \Box B$. Altogether this yields $u \Vdash t:B$. \dashv

To conclude the proof of Theorem 1, consider a Fitting S4B_2 -model, where $u \not\models F$. By the Main Lemma, $u \notin F^*$, hence F is not valid in the topological S4B_2 -model $\mathcal{T}_{\mathcal{K}}$. \dashv

2.5. Completeness with Respect to Real Topology

THEOREM 2. S4B_0 , S4B_1 , S4B_2 are complete with respect to the real topology \mathbb{R}^n .

PROOF. We will use the following main lemma from recent refinements of Tarski's Theorem from [12, 26, 34]:

LEMMA 3. *There is an open and continuous map π from $(0, 1)$ onto the Kripke topological space corresponding to a finite rooted Kripke frame.*

Such a map π preserves truth values of modal formulas at the corresponding points. It suffices now to refine the proof of Theorem 1 to produce a finite rooted Fitting counter-model for F and to define the test function $\mathcal{M}'(t, G)$ on $(0, 1)$ as

$$\mathcal{M}'(t, G) = \pi^{-1}\mathcal{M}(t, G).$$

The resulted topological model is a $(0, 1)$ -countermodel for F . This construction yields completeness with respect to the real topology \mathbb{R}^n , for each $n = 1, 2, 3, \dots$. \dashv

3. Future Work

The next natural steps in this direction could be introducing operations on tests. It also looks promising to introduce tests in systems of topological reasoning about knowledge [13] and Dynamic Topological Systems [7, 20].

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